Chirality and the isotopy classification of skew lines in projective 3-space

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Chiralité et l'isotopie des droites dans un espace projectif à 3 dimensions

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Résumé. Cet article traite des invariants d'isotopie des configurations des droites disjointes dans l'espace projectif à trois dimensions. Nous dévéloppons la théorie de la signature chirale d'une configuration, et du polynôme de Kauffman. L'invariance du polynôme de Kauffman par rapport à deux types de transformation est montrée par un argument combinatoire direct. Le lien entre une configuration et ses projections planes s'établit dans le contexte de la géométrie projective orientée, en se servant des lignes directrices orientées et des isotopies-limites. En passant au modèle sphérique de l'espace projectif, dû à Klein, nous obtenons un modèle de la configuration en anneaux enlacés, et à un modèle à boule et corde, dit modèle temari. Les graphes linéaires fournissent des codes pour les signatures chirales, et permettent l'identification des signatures qui ne se réalisent pas en tant que configuration empilée simple. Un catalogue s'ajoute, contenant les configurations non-etiquettées n'ayant pas plus que six droites, accompagnées de leurs polynômes de Kauffman.

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Abstract. This article concerns isotopy invariants of finite configurations of skew lines in projective 3-space. We develop the theory of the chiral signature and of the Kauffman polynomial of a configuration. Invariance of the Kauffman polynomial under two types of diagram moves is shown by a direct combinatorial argument. The connection between a configuration and its plane projections is established in the context of oriented projective geometry, using oriented directrices and limit isotopies. Using a map to the Klein spherical model of projective space, we arrive at a linked-circle model of the configuration, and to a convenient ball-and-string model, the temari model. Linear graphs provide codes for chiral signatures, and permit the identification of those signatures which can be realized as simple stacked configurations of lines, which we call spindles. A catalogue of all unlabeled configurations of up to six lines, together with their Kauffman polynomials, is appended.

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1. Introduction

Given two indexed families $L = \{L_1, \ldots, L_n\}$, and $M = \{M_1, \ldots, M_n\}$, each consisting of n mutually non-intersecting lines in real affine or projective 3-space, can the lines L_i of the family L be moved in a continuous manner, and in such a way that at no time do two of the lines intersect, until they arrive at the positions of the corresponding lines M_i, for $i = 1, \ldots, n$? If so, we say that the families L, M are *affinely* or *projectively isotopic*, respectively. There are how many distinct configurations of n lines, up to affine or projective isotopy? (The same questions can of course be asked in a context where the *order* of the lines is unimportant.)

The affine problem was brought to our attention by Gert Vegter (University of Groningen), who was looking for a non-trivial lower bound for the complexity of the following problem in computational geometry: *given n lines in 3-space, do two of them intersect*? Since the logarithm of the number of components of the configuration space for skew lines provides such a lower bound, he had become interested in the isotopy classification problem. A glance at the affine problem shows that it is trivial. In Section 6 we give our proof that there is only one affine isotopy class. The projective problem, however, turns out to be a lot more interesting. Even for $n = 3$ there are two families which are not projectively isotopic. To the present day, there are known only three isotopy invariants, these providing a partial classification of isotopy classes of projective line configurations: their *chiral signatures* (Section 3) and their *Kauffman polynomials* (Section 13), and the Morton trace of the associated braid [Pe_6].

Our approach to the isotopy classification of projective line configurations can be characterized as constructive, algebro-geometric, and 3-dimensional. Constructive, in the sense that we employ explicit geometric constructions rather than proofs of existence. Algebro-geometric, in that we calculate with the Grassmann-Plücker coordinates of flats in projective space, rather than employ topological arguments. We prefer to describe isotopies by specific linearly parametrized projective maps. In so doing, we try to bridge the gap between pertinent geometric concepts and their realization in terms of computer algorithms adequate to test hypotheses involving those concepts. Finally, by working directly in 3-dimensional space, we can sometimes achieve a higher degree of invariance in the statement and proofs of theorems, side-stepping the necessity to provide proofs that certain properties of line configurations, albeit measured in plane projection, are in fact independent of projection.

Initially, we work with unoriented lines. This is adequate (indeed preferable) for a discussion of chiral invariants. We define the *chiral signature* of a configuration of n lines, an arithmetic invariant which associates signs $\{\pm 1\}$ to all 3-element subsets of an n-element set. The crucial property (indeed, the only known general property) of chiral signatures is that they satisfy the *rule of four*, that is, that the product of the chiralities of the four triples drawn from any set of four of the lines is equal to $+1$. By introducing a *graph-theoretic notation* for chiral signatures, we define an equivalence relation "is a cousin of" on $(n-1)$-vertex graphs, in such a way that each set of (unlabelled) "cousins" specifies a unique (unlabelled) chiral signature. This procedure permits a rapid visual recognition of individual chiral signatures. Using this

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graph-theoretic notation, we have computed all abstract chiral signatures (sign functions satisfying the rule of four) for up to 8 lines, and have shown (by hand) that *all* are chiral signatures of configurations of lines.

Other properties of line configurations are more easily stated in terms of *oriented* lines. and the linking numbers of pairs of such lines. We give a brief introduction to oriented projective geometry in Section 4. This topic is the subject of a recent work by J. Stolfi [St]. In the context of oriented projective geometry, there are two crucial levels of concretization of line configurations: (1) to consider the configuration in perspective, from a certain point of view z somewhere in $\mathcal{P}^3$, and (2) to scan the line configuration with respect to a line K through z. The perspective view of a line configuration gives rise to the combinatorial structure of a *link chirotope*(Section 8), while the scan of the configuration with respect to a line through the center of perspective yields a *geometric braid*(Section 12). These intermediate combinatorial structures, richer than the chirality type of a configuration, poorer than the configuration itself, provide the essential "ladder" for the study of geometric realizations of "abstract" line configurations.

For most purposes, line configurations are best studied in plane projection. Since projected lines are *coplanar*, a certain "fiction" is usually introduced to retain crossover information in the projected image. We choose instead to obtain this information as the *projective limit* of an isotopy which carries line configurations (in the limit) *to* their images under arbitrary projections. In this way, the "fictitious" quantities are replaced by legitimate non-zero values. We can equally well demonstrate the existence of "nearby" 3-dimensional models which live in ϵ-height cylindrical neighborhoods of a circular domain containing all crossing points in the projection plane, models which clearly realize the crossing data presented in projection. These concepts are presented in Sections 5-7. We show how to compute all invariants both in $\mathcal{P}^3$, and in $\mathcal{P}^2$, directly from any projection.

It is always possible to represent any given line configuration L in a *regular* projection, where no projected line $A' = \pi(A)$ degenerates to a point, where no two projected lines A', B' are parallel, and where no three projected lines A', B', C' are concurrent. But it is not possible to avoid singularities of projection when one uses a single projection to follow the action of an isotopy of lines. In section 9, we analyze the three basic types of singularities.

To each line configuration L, no line L of which is at infinity, we associate a specific configuration of linked geometric circles $\gamma(L)$ in $\mathcal{R}^3$. These circles $\gamma(L)$ lie in planes through the origin, and intersect the unit sphere at antipodal points. This geometric model $\gamma(L)$ is the *Klein link* of the line configuration. The Klein link is constructed by covering $\mathcal{P}^3$ by S^3 and applying a stereographic projection from S^3 to $\mathcal{R}^3$. In Section 10 we give a complete algebraic and geometric description of the circles that are the components $\gamma(L)$ of the Klein link. We may then choose to *forget* the geometric structure, and to retain only the topological structure of the (*flexible*) link. In so doing, we lose significant geometric information (the straightness of lines L, the circularity of $\gamma(L)$ and its relation to the unit sphere), but we gain, in return, powerful methods of analysis (methods of calculation of link polynomials, which force us to depart from the context of geometric Klein links).

If an (oriented) planar layout is mapped by central projection onto a 2-sphere, one obtains a configuration of (oriented) great circles. By a limit isotopy of the Klein model (or indeed, by a limit isotopy of the planar layout itself) we can maintain cross-over information, and thus construct a model imagined as a solid ball covered by *mutually linked* great circles of string. We call these balls *temari models* (Section 11).

During the preparation of this paper we got acquainted with the work of O.Ya. Viro and, recently, of V.F. Mazurovskii. Viro proved that projective configurations of up to 5 lines are characterized by their chiral signatures [Vi], while Mazurovskii showed that the Kauffman polynomial distinguishes all projective configurations of at most 6 lines [Ma_1]. Independently from our work, they recognized the importance of associating a *link* with a configuration of lines. In his approach, Mazurovskii regards the lines as closed curves in $\mathcal{P}^3$ and thus obtains a link directly in $\mathcal{P}^3$. At that point in his argument he appeals to the results of Yu.V. Drobotukhina on the isotopy of links in $\mathcal{P}^3$ and her generalization of Reidemeister's theorem by means of diagrams in the projective plane [Dr]. We do not make use of Drobotukhina's theory, preferring to substitute the direct geometric construction of the Klein model, and its limit, the temari model.

In Section 12 we provide an easy way to construct an associated planar *link diagram*, starting from a planar layout of a given configuration. We then establish the connection between this link diagram, the Klein model, and the temari model of the configuration.

It is conceivable that *flexible* isotopies are more general than *rigid* isotopies. That would mean that there are distinct isotopy classes of line configurations which are topologically indistinguishable (being isotopic as links). To find such an example (or to show that none exists) is perhaps the most significant unsolved problem in this area. As a first step towards the construction of such examples we give (Section 13) a sequence of intermediate relaxations between the *rigid* isotopy of lines in $\mathcal{P}^3$ and the *flexible* isotopy of Klein links in $\mathcal{R}^3$. To each planar layout is associated a stretchable *pseudoline diagram* (where the lines are allowed to be curved, but still intersect each other exactly once, etc.). Each such diagram is also simply an arrangement of pseudolines, forgetting the restriction that it be stretchable. And finally, it may be completed to its corresponding link. At each stage we apply two types of *diagram moves* to generate isotopies, in the same spirit as the Reidemeister moves for knot diagrams (Section 14). But as the context becomes progressively freer, in the passage from planar layouts to links, we risk finding ourselves using isotopies of diagrams which do not lift to isotopies of spatial configurations, or using sequences of intermediate line diagrams that differ by diagram moves but that cannot be connected by a continuous homotopy of lines in the plane, or even passing from a stretchable pseudoline diagram to another by way of a non-stretchable one. It is thus conceivable that the freer contexts, such as links, have significantly fewer isotopy types. Any given such type may split into several isotopy types of line configurations.

We give a direct proof that a *Kauffman polynomial* computed on planar layouts in the projective plane is an invariant under diagram moves (Section 14). Our independent calculation of these polynomials confirms the coefficients given for these polynomials in [Ma$_3$], but does not go so far as to confirm that his list is exhaustive. So far, Mazurovskii's progressive generalization of degenerate line configurations in the only proof of that classification. A table of unlabelled configurations of up to 6 lines, together with the graphs of their chiral signatures and their Kauffman polynomials, will be found in Appendix A.

In Section 15 we examine some special class of configurations, called *spindles*, where the lines all intersect some fixed pair of skew lines. We observe that any spindle can be moved to a configuration where the lines are "stacked" in horizontal planes. We give sufficient conditions under which two spindles are isotopic, using *spindle moves*. Since most chiral signatures are representable by spindles, it is of some interest to know which graphs denote chiral signatures *minimal* with the property that they are not spindle-representable. We provide that information in Section 16.

Before we turn to our first topic, calculations of Grassmann-Plücker coordinates in projective geometry, we give a strict topological definition of isotopy of line configurations. It is convenient to employ the one-to-one correspondence between lines in $\mathcal{P}^3$ and 2-dimensional subspaces of $\mathcal{R}^4$, which are *points* on the Grassmannian $G_{2,4}$, itself, a quadratic hypersurface in $\mathcal{P}^5$. The condition that two lines are incident in $\mathcal{P}^3$ gives a quadratic relation in their Plücker coordinates. So a configuration of n mutually skew lines can be considered as a point in the complement of $\binom{n}{2}$ quadratic surfaces with respect to the product space $(G_{2,4})^n$, and hence as a point in an open submanifold $\mathcal{C}$ of $(G_{2,4})^n$. An isotopy of lines is simply a continuous path between two points in $\mathcal{C}$. When one prefers to study the isotopy of unlabelled configurations, one should take the quotient of $\mathcal{C}$ under the action of the symmetric group S_n.

2. Projective calculations

Algebraic calculations in projective geometry are performed using Grassmann-Plücker coordinates and the Cayley algebra of subspaces [Do]. Before turning to the projective isotopy problem, we give a brief exposé of these methods.

The $(k-1)$-dimensional flats (lines, planes, ...) in $\mathcal{P}^{n-1}$ are the rank k subspaces of $\mathcal{R}^n$. Such flats are of *rank* k. At the ends of this spectrum of flats are found the 0-flats, or *scalars*, and the n-flats, or *pseudo-scalars*, both of which have only a single coordinate, one real number. Projective points are *coordinatized* (uniquely, up to a non-zero scalar multiple) by any non-zero vector in their associated 1-dimensional subspace. Whenever we perform a calculation involving a projective point p, we make a specific choice of coordinates for p, and maintain that choice for the remainder of the calculation.

A set of projective points is said to be *independent* if and only if they have coordinate vectors which are linearly independent. The Grassmann-Plücker coordinates of a flat A are obtained, up to a non-zero scalar multiple, as the $k \times k$ minors of a matrix whose rows are coordinate vectors for k independent points on the flat A. Each component of such a coordinate vector has a *label* or *place*, namely, the k-element set of columns $\{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$ used to form the minor in question, and listed in increasing order. When displaying the coordinates of a flat, we either explicitly display the labels, or simply list the coordinates in the lexicographic order of their places.

In order to have a reasonable mental image of a configuration of finitely many projective lines, pairwise skew in $\mathcal{P}^3$, we will invariably work with a corresponding affine model. To this end, we choose as *plane at infinity* some plane I which is transverse to all lines in the configuration. (This can always be done, because the set of planes containing at least one of the lines is itself the union of n one-dimension pencils in the 3-dimensional space of all planes, the Grassmannian $G_{3,4}$.) We then select a coordinate system in which the equation of the plane I is $x_4 = 0$. Points p not on the plane I, which we shall call *finite* points with respect to this choice of plane at infinity, have coordinate vectors (p_1, p_2, p_3, p_4) with $p_4 \neq 0$, and can be associated with the Euclidean point p_{euc} with coordinates $(\frac{p_1}{p_4}, \frac{p_2}{p_4}, \frac{p_3}{p_4})$. The finite points thus form a Euclidean space $\mathcal{R}^3$ within $\mathcal{P}^3$. Finite points are uniquely representable in *standard* form, with 4^{th} coordinate equal to 1.

Let N be the index set $N = \{1, 2, \ldots, n\}$ of coordinate places for points. For any value of k, $0 \leq k \leq n$, let $\binom{N}{k}$ be the set of ordered k-element subsets of N, in *increasing* order. For any two *disjoint* subsets $I \in \binom{N}{j}, J \in \binom{N}{k}$, define

$$sign\ (I, J) = +1 \text{ or } -1,$$

takes "I then J" (both sets separately in increasing order, with I before J) to the union "$I \cup J$" (merged in increasing order). We shall use "square" cups and caps to indicate disjoint union and the complementary notion, co-disjoint intersection:

$$\begin{aligned} I \sqcup J = K &\iff I \cup J = K \text{ and } I \cap J = \emptyset, \\ I \sqcap J = K &\iff I \cap J = K \text{ and } I \cup J = N. \end{aligned}$$

There are three principal operations in the algebra of flats: join ($\vee$), meet ($\wedge$), and complement (*), analogues of the operations ($\cup, \cap, \sim$) of Boolean algebra. These operations are easily defined in terms of the Grassmann-Plücker coordinates. The *join* of a flat S of rank i with a flat T of rank j is the $(i+j)$-flat $S \vee T$ with coordinates

$$(S \vee T)_K = \sum_{I \sqcup J = K} sign\ (I, J)\ S_I T_J, \tag{1}$$

the sum being over all expressions of K as the disjoint union of subsets I and J of cardinality i and j, respectively. In terms of subspaces, the join of two flats is the linear span of their union.[2] The join of a

[2] This is the operation "wedge" of exterior algebra, usually denoted "$\wedge$". We prefer the notation "$\vee$" of Cayley algebra, as an aid to geometric intuition.

set of points, we abbreviate to a simple concatenation: $p \vee q \vee r$ becomes pqr. The join of any *dependent* set of points is equal to 0.

The *meet* of a flat S of rank i with a flat T of rank j is the $(i+j-n)$-flat $S \wedge T$ with coordinates

$$(S \wedge T)_K = \sum_{I \cap J = K} \mathit{sign}\ (I \backslash K, J \backslash K)\ S_I T_J, \tag{2}$$

the sum being over all expressions of K as the co-disjoint intersection of subsets I and J of cardinality i and j, respectively. (The symbol $\backslash$ denotes set-theoretic difference.)

As a parenthetical remark, if $\vee$ and $\wedge$ are regarded as operations on subspaces of the vector space $\mathcal{R}^n$, then for subspaces S and T, the join $S \vee T$ is the *linear span*

$$\langle S, T \rangle \;=\; \{as + by;\ a, b \in \mathcal{R}, s \in S, t \in T\},$$

when S and T have no non-zero vectors in common, and the meet $S \wedge T$ is the *intersection* $S \cap T$ whenever $\langle S, T \rangle \;=\; \mathcal{R}^n$.

The *complement* of a flat S of rank k is the flat S^* of rank $n-k$ with coordinates

$$(S^*)_{N \backslash I} = \mathit{sign}\ (I, N \backslash I)\ S_I. \tag{3}$$

Two flats are complementary if and only if their associated subspaces are orthogonal complementary, relative to inner product formed using the standard basis. This operation "*" is usually called the *Hodge star complement.* It satisfies identities analogous to those in Boolean algebra:

$$\begin{aligned} (S \vee T)^* &= S^* \wedge T^* \\ (S \wedge T)^* &= S^* \vee T^* \\ S^{**} &= (-1)^{k(n-k)} S \end{aligned} \tag{4}$$

The complement is extremely useful for expressing orthogonality relations. For instance, if o is the origin and L is a line, then

$$o \vee L^*$$

is the plane through o perpendicular to L, while

$$(o \vee L^*) \wedge L$$

is the perpendicular projection of o on L, that is, the closest point to o on the line L.

The *bracket* of n points $a, b, \ldots, e$ in a projective space of rank n, written $[ab \ldots e]$, is the determinant of the $n \times n$ matrix of their coordinates. Any bracket $[ab \ldots e]$ is equal to the repeated join of those points, $a \vee b \vee \ldots \vee e$. From this it follows that any bracket with *repeated* or *dependent* points is zero. Thus $[abad]$ in zero for all points a, b, d in $\mathcal{P}^3$, while $[abcd]$ is zero when, for instance, a, b, d are collinear, or, more generally, when a, b, c, d are coplanar.

The complement of a hyperplane H (a flat of rank $n-1$) provides the vector of coefficients of the equation for H. We see this as follows. A point x lies on H if and only if the bracket satisfies the equation

$$\begin{aligned} [xH] &= x \vee H \;=\; \textstyle\sum_{i=1}^{n} \mathrm{sign}\,(i, 1 \ldots \hat{i} \ldots n)\ x_i H_{1 \ldots \hat{i} \ldots n} \\ &= (-1)^{n-1} \textstyle\sum_{i=1}^{n}\ x_i H_i^*, \end{aligned}$$

because

$$\mathrm{sign}\,(i, 1 \ldots \hat{i} \ldots n) \;=\; \mathrm{sign}\,(1 \ldots \hat{i} \ldots n, i)\ (-1)^{n-1}.$$

For flats given as joins of points, the bracket can be used to express their meet. The signs of the additive terms are the signs *sign* (A,B) of the *splits* $S = A \sqcup B$ or $T = A \sqcup B$ of the ordered sets S or T of points into two parts, one of which is the right size to "fill" the bracket:

$$S \wedge T = \sum_{A \sqcup B = T} sign\ (A,B)\ A[SB] = \sum_{A \sqcup B = S} sign\ (A,B)\ [AT]B \tag{5}$$

This identity was used in [Do] to define meet in terms of bracket and join, as the main axiom of the *Cayley algebra* of projective flats.

The minors of any rectangular matrix obey certain polynomial equalities. Consequently, there exist polynomial relations, called *p-relations*, among the Grassmann-Plücker coordinates of the space spanned by k points in a space of rank n. We shall have occasion to use the p-relation among the six coordinates of a line L in 3-space (rank 4):

$$L_{12}L_{34} - L_{13}L_{24} + L_{14}L_{23} = 0. \tag{6}$$

Likewise following from the polynomial relations among minors of a rectangular matrix, are the *generic first-order syzygies*. These are expressible in two forms: either as linear relations (with brackets as coefficients) among $n+1$ points in rank n, or as polynomials in brackets which invariably evaluate to zero. We shall make use of the first order syzygy among 5 points a, b, c, d, e in rank 4:

$$[abcd]e - [abce]d + [abde]c - [acde]b + [bcde]a \ = \ 0 \tag{7}$$

and of the corresponding bracket polynomial identity

$$[abcd][efgh] - [abce][dfgh] + [abde][cfgh] - [acde][bfgh] + [bcde][afgh] \ = \ 0 \tag{8}$$

obtained by joining equation (7) with a plane $Q = fgh$.

The following theorem of Cayley algebra is indispensible for any serious work in projective 3-space. See Figure 1. Note that $(z \vee Q) \wedge L$ is a scalar multiple of the line L itself.

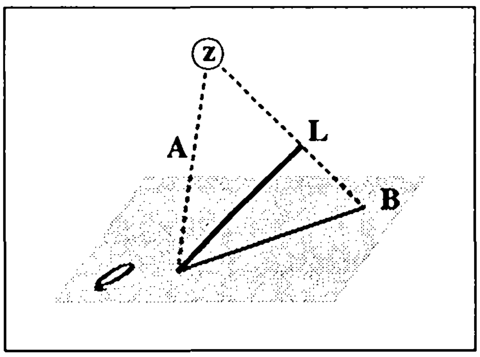

Figure 1. The lines $A = z \vee (L \wedge Q)$, L, and the projection $B = (z \vee L) \wedge Q$ lie in a flat pencil, and are dependent.

Theorem 1. *For any point z, any line L, and any plane Q in $\mathcal{P}^3$,*

$$z \vee (L \wedge Q) \ = \ (z \vee Q) \wedge L \ - \ (z \vee L) \wedge Q.$$

Proof: Let $z = a,\ L = bc,\ Q = def$. Then

$$\begin{aligned} z \vee (L \wedge Q) \ &= \ a \vee (bc \wedge def) \ = \ ac[bdef] \ - \ ab[cdef] \\ (z \vee Q) \wedge L \ &= \ [adef]bc \\ (z \vee L) \wedge Q \ &= \ abc \wedge def \ = \ [adef]bc \ - \ [bdef]ac \ + \ [cdef]ab \end{aligned}$$

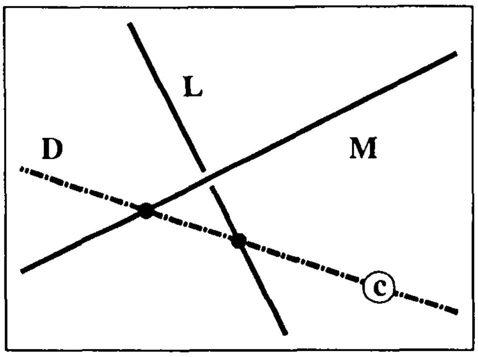

Figure 2. Example of projective calculations

So the stated equality holds. □

As an example of the calculations mentioned in this section, say we wish to find the line D passing through a point c, and meeting two skew lines L and M. The desired line, as in Figure 2, is given by the expression

$$D \;=\; (c \vee L) \wedge (c \vee M).$$

Say c is at height 1 on the z-axis, L is a vertical line passing through the Euclidean point $(1,3,0)$, and M is the line at infinity in the plane Q with equation $3x + 7y - 4z = 2$. The projective calculation proceeds as follows.

$$\begin{array}{rcl}
c & = & (0\;\;0\;\;1\;\;1) \\
L & = & (1\;\;3\;\;0\;\;1) \vee (0\;\;0\;\;1\;\;0) = (0\;\;1\;\;0\;\;3\;\;0\;\;-1) \\
Q & = & (3\;\;7\;\;-4\;\;-2)^* = (2\;\;-4\;\;-7\;\;3) \\
M & = & (2\;\;-4\;\;-7\;\;3) \wedge (1\;\;0\;\;0\;\;0) = (4\;\;7\;\;0\;\;-3\;\;0\;\;0) \\
c \vee L & = & (0\;\;0\;\;1\;\;3) \\
c \vee M & \sim & (4\;\;4\;\;7\;\;-3) \\
D & = & (0\;\;-4\;\;-4\;\;-12\;\;-12\;\;-24) \sim (0\;\;1\;\;1\;\;3\;\;3\;\;6)
\end{array}$$

where by "$L \;=\; (0\;\;1\;\;0\;\;3\;\;0\;\;-1)$", we mean

$$L \;=\; \begin{pmatrix} \mathbf{12} & \mathbf{13} & \mathbf{14} & \mathbf{23} & \mathbf{24} & \mathbf{34} \\ 0 & 1 & 0 & 3 & 0 & -1 \end{pmatrix}$$

and by "$Q \;=\; (2\;\;-4\;\;-7\;\;3)$" we mean

$$Q \;=\; \begin{pmatrix} \mathbf{123} & \mathbf{124} & \mathbf{134} & \mathbf{234} \\ 2 & -4 & -7 & 3 \end{pmatrix},$$

the places always being listed in lexicographic order.

3. Chiral Invariants of Projective Isotopy

We turn to the problem of classifying configurations of projective lines. From now on, a *configuration* will always stand for a set of n mutually skew lines in $\mathcal{P}^3$. Without loss of generality, such configurations are representable in $\mathcal{R}^3$, if we make sure that no line is in the plane at infinity of the projective completion of $\mathcal{P}^3$. Our first task is to give a projectively invariant treatment of *chiral signature*.

Let p, q be distinct points on a line A in projective 3-space. The line L is then the join $p \vee q$ of those points, and a set of coordinates for L is furnished by the minors of the 2×4 matrix whose rows are coordinate vectors $(p_1, \ldots, p_4)$, $(q_1, \ldots, q_4)$ for p and q:

$$A = (A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}), \text{ where } A_{ij} = p_i q_j - p_j q_i.$$

The operation $\vee$ is anticommutative on points: $p \vee q = -q \vee p$.

Given two disjoint (that is, skew) lines in projective 3-space, we calculate the join

$$A \vee B = A_{12}B_{34} - A_{13}B_{24} + A_{14}B_{23} + A_{23}B_{14} - A_{24}B_{13} + A_{34}B_{12},$$

a pseudoscalar quantity which, for lines not lying on the plane at infinity, can be thought of as (six times) the volume of a tetrahedron formed by two points (in *standard* coordinates) on A, and two on B, such that $A = a_1 a_2, B = b_1 b_2$. Acting on lines in 3-space, the operation "join" is bilinear and *symmetric*: for any scalar k,

$$kA \vee B = A \vee kB = k(A \vee B), \text{ while } A \vee B = B \vee A.$$

The join $A \vee B$ is 0 if and only if the lines A and B have a projective point in common, or equivalently, if and only if they are coplanar.

For any scalar or pseudoscalar quantity s, let $\textsf{sign}\,(s)$ denote the sign, $+1$ or -1, of the real number s. Given three mutually skew lines A, B, C in projective 3-space, with fixed (but arbitrary) coordinates, we form the product of three pseudoscalars to obtain

$$\textsf{vorticity}\,(A, B, C) = (A \vee B)(A \vee C)(B \vee C),$$

the *vorticity* of the triple of lines. Given a family of n lines in projective 3-space, its vorticity is the function which assigns to every triple ABC of lines the value $\textsf{vorticity}\,(A, B, C)$. The *chiral signature* (or simply the *chirality*) χ of a family of lines is the function which assigns to every triple ABC of those lines the sign

$$\chi_{ABC} \;=\; \textsf{sign}\,(\textsf{vorticity}\,(A, B, C))$$

of the vorticity of those lines. The next two theorems deal with the geometric significance and projective invariance of this concept.

Theorem 2. *The vorticity of a family of lines is a projective invariant. The chirality of a family of lines is an isotopy invariant. It is symmetric, and orientation-invariant, in the sense that multiplication of the projective coordinates of any one line by a scalar (positive or negative) does not change its value.*

Proof: Since the vorticity of a triple of lines is expressible as a product of determinants of projective points, it is a projective invariant. Since the operation "join" is symmetric on lines in 3-space, in that $A \vee B = B \vee A$, permutation of the lines in a triple does not change its signature. The operation "join" is also multilinear, so if we multiply the coordinates of one of the lines A, B, C by a scalar s, the vorticity is multiplied by s^2, and its sign is unchanged. Consequently, the chirality of a family is completely independent of the manner in which the lines are coordinatized. In particular, it is orientation-invariant.

The projective coordinates of lines, their joins, and the resulting calculation of vorticity are continuous functions of the positions of the lines and of the overall scalar multiples used in their coordinatization. The vorticity $\textsf{vorticity}\,(A, B, C)$ of a triple of lines is zero if and only if one of the three terms of the product $(A \vee B)(A \vee C)(B \vee C)$ is zero, if and only if some pair of the three lines intersect at a point. Thus, under an isotopy of a family of lines, there can be no change in the sign of the value χ_{ABC} for any triple. The chirality is isotopy invariant. □

Theorem 3. *For any four relatively skew lines A, B, C, D in projective 3-space, the product*

$$\chi_{ABC}\ \chi_{ABD}\ \chi_{ACD}\ \chi_{BCD}$$

is positive.

Proof: The stated product is the sign of a product

$$(A \vee B)^2 (A \vee C)^2 (A \vee D)^2 (B \vee C)^2 (B \vee D)^2 (C \vee D)^2,$$

of squares of joins of lines. Each join is non-zero because the lines are skew, so the product of squares is strictly positive. □

Theorem 4. *The chirality of three lines A, B, C, is positive if and only if, looking down any one of the lines (in either direction), then proceeding outward toward the apparent point of intersection of the other two lines, the line on the left during this approach passes over the line on the right.*

Proof: Assume that the plane at infinity I has been chosen such that it contains no line of the given configuration. Represent finite points by standard projective coordinates, with 4^{th} coordinate equal to 1. Taking points q, p in order along the direction of projection along the line A, points r, s in order on the left-hand line, approaching the apparent point of intersection, and points t, u in order on the right-hand line, departing from the apparent point of intersection, all three 4 by 4 matrices of coordinates in the expression

$$[p, q, r, s]\ [p, q, t, u]\ [r, s, t, u] = (A \vee B)(A \vee C)(B \vee C) = \chi_{ABC}$$

have positive determinants.[3] □

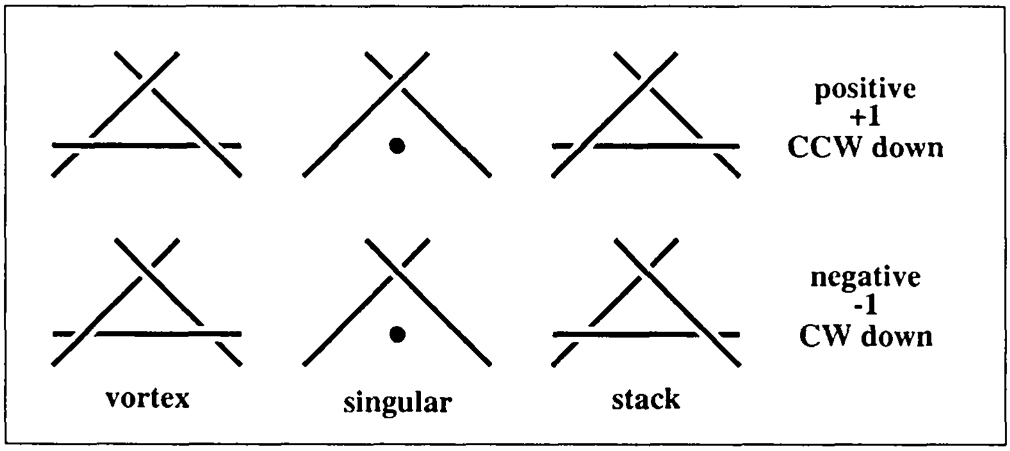

Figure 3. Chirality of three lines, as seen in plane projection

[3] For such calculations, it helps to remember that a determinant $[pqrs]$ is positive for points expressed in standard coordinates if and only if the point p lies on the positive side of the oriented triangle qrs, this being decided by the "right-hand rule": the fingers of the right hand give the cyclic order qrs in their plane, while the thumb points out of that plane, toward p.

Figure 3 illustrates the various possible plane projections of a triple of lines, and indicates the corresponding value of the chirality. In the *vortex* form of the projected image, a *positive* triple of lines appears to spiral *counter-clockwise* downward into the page. We can rotate one of the lines in a vertical plane, about the midpoint of the indicated segment as center, without causing two lines to meet. After a quarter-turn, the line is vertical, as shown in the *singular* projection. In this position, we can judge the chirality of the triple by reference to Theorem 4. Continuing the same rotation for somewhat less than another quarter-turn, we arrive at a configuration in which the projected lines seem to be *stacked*. The stacking order, going downward into the page, is in the *counter-clockwise* order of the lines around the triangular region they bound. For a *negative* triple of lines, the corresponding figures show a *clockwise* rotation down into the page.

For any n-element set L, an *abstract signature* on L is any function

$$\alpha : \binom{L}{3} \to \{\pm 1\}$$

satisfying the conclusion of Theorem 3, the *rule of four*. Fix an arbitrary element X in L as *root*, and define a corresponding graph $G_X(\alpha)$ to have vertex set $L \backslash \{X\}$ and edge set

$$\{(A,B);\ A,B \in L \text{ and } \chi_{XAB} = -1\}$$

With respect to different roots, a single signature can give rise to a number of substantially different graphs. For instance, the abstract signature

ABC	ABD	ABE	ABF	ACD	ACE	ACF	ADE	ADF	AEF
-1	$+1$	$+1$	$+1$	-1	$+1$	$+1$	-1	$+1$	$+1$
BCD	BCE	BCF	BDE	BDF	BEF	CDE	CDF	CEF	DEF
$+1$	-1	-1	-1	$+1$	$+1$	$+1$	-1	$+1$	-1

gives rise to graphs in three distinct isomorphism classes, for roots B, D, F, as in Figure 4.

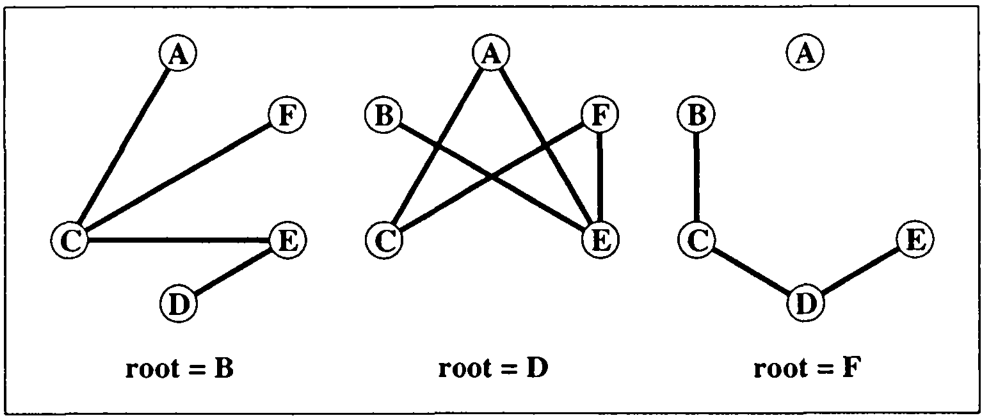

Figure 4. Three cousins, graphs of a single signature, with respect to roots B, D, F, respectively.

In order to compare the different graphs arising from a single chiral signature, we introduce the following purely graph-theoretic notions. Two graphs on the same set V of vertices are *brothers* if and only if the symmetric difference of their edge sets is a complete bipartite graph on the vertex set V. For any graph G, let G^+ be the graph G together with a single isolated vertex. Two unlabelled graphs G, H, are *cousins* if and only if the graphs G^+, H^+ are brothers (under some correspondance between the vertex sets of G^+, H^+).

Using a list of all isomorphism types of graphs on up to 7 vertices, provided by Professor Katsuhiro Ota, we determined the partition of this set of graphs into sets of cousins. The number of distinct (cousin) equivalence classes of graphs is given by the following table. The row indicates the number of lines in the configuration, the column, the number of triples with negative chirality, and the table entry, the number of distinct classes of cousins yielding a chiral signature with those characteristics.

$$3: \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad 4: \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix} \quad 5: \begin{bmatrix} 0 & 3 & 4 & 5 & 6 & 7 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$6: \begin{bmatrix} 0 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 20 \\ 1 & 1 & 1 & 3 & 4 & 3 & 1 & 1 & 1 \end{bmatrix}$$

$$7: \begin{bmatrix} 0 & 5 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & \ldots \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 4 & \ldots \end{bmatrix}$$

$$8: \begin{bmatrix} 0 & 6 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 29 & 30 & \ldots \\ 1 & 1 & 1 & 2 & 1 & 4 & 7 & 9 & 17 & 26 & 31 & 43 & 31 & 26 & \ldots \end{bmatrix}$$

The dots in the tables for seven and for eight lines indicate symmetry of the table about the middle binomial coefficients, 17/18 and 28, respectively. For any *even* number of lines, it is possible that a class of cousins has chirality half-positive, half negative. Then there are two possibilities: each graph in that class may be cousin to its own complement, or not. For four and for six lines, the former is always the case, but for eight lines this is no longer so. Among the 43 classes of cousins with chirality half-positive, half-negative, there are 19 cases in which each graph is cousin to its own complementary graph, and $12 \times 2 = 24$ cases in which this is not so.

Theorem 5. *All abstract signatures on an n-element set are realizable as the chiral signature of some set of n skew lines in $\mathcal{P}^3$, for $n \leq 8$.* □

Theorem 5 was settled by hand calculation, for each class of cousins. Where the representation could not be given in terms of stackings of lines, or *spindles* (see Section 14), we used a simple shelling principle to prove that each line diagram was in fact realizable by straight lines in space. We had rather expected that non-realizable abstract signatures would arise before $n = 8$. This not being the case, we tried for a while to prove that all abstract signatures are realizable. This is *not* so. Peter Shor has provided us with a non-realizable abstract signature on 71 lines. He constructed this example using the simpler idea that 4 planes cannot divide real 3-space into 16 cells.

4. Oriented lines in point perspective

In this section, we make a detailed study of two oriented lines. We observe such a pair of lines from a single point in space, that is, in point perspective.

We construct an *oriented projective space* OP^n from $\mathcal{R}^{n+1}$ by identifying vectors which differ only by a *positive* scalar multiple. A *flat* in OP^n is then an equivalence class of vector representations (Plücker coordinates) of projective flats (points, lines, planes, ...) under positive scalar multiplication[4]. For a detailed

[4] Every finite point a is "really" a pair of antipodal points $a, -a$ on S^n, superimposed by central projection onto a hemi-n-sphere. The lines $a \vee b$ and $-a \vee b$ have opposite orientation.

treatment of oriented projective space, we refer the reader to Jorge Stolfi's recent doctoral dissertation [St] and subsequent book[5] on the subject.

We restrict our attention to OP^3. As usual, homogeneous coordinates with last coordinate x_4 equal to zero represent points at infinity. An oriented point a is said to be *positive* if and only if its fourth coordinate a_4 is positive. Notice that a positive point is always finite. Under the *standard* embedding of $\mathcal{R}^3$ in OP^3, each euclidean point is mapped to a positive point:

$$(a_1, a_2, a_3) \in \mathcal{R}^3 \rightarrow (a_1, a_2, a_3, 1) \in OP^3.$$

We say $\mathbf{sign}\,(a) = \mathbf{sign}\,(a_4)$ is the *sign* of the point a.

Let L be an oriented projective line and take two oriented points a and b on L. Since $0 \neq a \vee b = -b \vee a$ we are free to define $a \vee b$ as the line L *oriented from* a *to* b. If L is not on the plane at infinity and if a and b are positive points then we can represent the orientation $a \vee b$ visually by a line segment with arrow head pointing from a toward b.

If A and B are coordinatizations of the *same* projective flat, then $A = wB$ for some non-zero scalar w, and we can define the *relative orientation* $\omega(A, B)$ to be $\mathbf{sign}\,(w)$ of the scalar w. In particular, for any distinct oriented points a, b on an oriented projective line L, $\omega(a \vee b, L)$ has value $+1$ or -1 accordingly as the line L is oriented from a to b, or from b to a.

A projectively-invariant definition of orientation is obtained by specifying a cyclic order of three points a, b, c on a line L. We say L is *oriented in the cyclic order* (abc) if and only if the product

$$\mathbf{relOrient}\,(L, (a, b, c)) = \omega(a \vee b, L)\ \omega(a \vee c, L)\ \omega(b \vee c, L)$$

is positive. The sign $\mathbf{relOrient}\,(L, (a, b, c))$ is called the *relative orientation* of a triple (a, b, c) of points on the line L. Observe that this definition does not depend on the orientation of the points a, b, c.

We let the oriented point $(0, 0, 0, 1)$ represent the origin, and choose an orientation of the plane I at infinity which makes $[oI] = +1$, that is,[6]

$$I = (-1, 0, 0, 0).$$

For any oriented line L not lying in the plane at infinity, its *direction* d_L is the oriented point

$$d_L = L \wedge (-1, 0, 0, 0)$$

where it meets the plane at infinity.

Proposition 6. *With these choices for o and I, the ray through the origin and the Euclidean point $a_{euc} = (a_1, a_2, a_3)$ has direction $(a_1, a_2, a_3, 0)$.*

Proof:

$$(o \vee a) \wedge I = (0, 0, -a_1, 0, -a_2, -a_3) \wedge I = (a_1, a_2, a_3, 0).$$

□

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[5] In our opinion, Stolfi's main contribution in that article was not his (correct) definition of the operations of oriented projective geometry; these are more clearly and simply specified in algebraic form (Cayley algebra), as shown at the end of his paper. Rather, his contribution was to give a coherent and geometric interpretation of the signed quantities occurring in such computations. Oriented projective geometry stands at "mi-chemin" between projective geometry and Cayley algebra. The latter is not only "oriented", but "sized" (or "weighted").

[6] There is no satisfactory way of "getting rid of" this minus sign when working in spaces of odd dimension (even rank). In general, $I = ((-1)^{n-1}, 0, 0, \ldots)$.

Proposition 7. *For any positive point a on a directed line L,*

$$\omega(a \vee d_L, L) = +1.$$

Furthermore, the join $L = a \vee b$ of two positive points will be oriented in the cyclic order (abd_L), where d_L is the direction of the line L.

Proof: If a is a positive point on a line L and $I = (1,0,0,0)$ is the plane at infinity, then $a \vee L = 0$, so by Theorem 1 of Section 2,

$$a \vee d_L \;=\; a \vee (L \wedge I) \;=\; (a \vee I) \wedge L \;=\; a_4 L,$$

a positive scalar multiple of L, so $\omega(a \vee d_L, L) = +1$. If L is the join $L = a \vee b$ of positive points a, b, then $\omega(a \vee d_L, L)$ and $\omega(b \vee d_L, L)$ are both equal to $+1$, and

$$\mathbf{relOrient}\,(L, (a, b, d_L)) \;=\; \omega(a \vee b, L) \;=\; +1.$$

□

The two previous propositions justify our choice for the orientation of the plane at infinity. An oriented point $p = (p_1, p_2, p_3, 0)$ at infinity is conveniently visualized as a ray emanating from the origin, passing through the standard image $q = (p_1, p_2, p_3, 1)$ of the Euclidean point $p_{euc} = (p_1, p_2, p_3) \in \mathcal{R}^3$. The point p is the direction of the ray $o \vee q$. The notion of oriented line, with its visual representation, extends equally well to the case where L is a line lying within the plane at infinity. To this end, we use the circle Γ_L of intersection of the plane $Q_L = o \vee L$ through the origin and through L, with the sphere S of unit radius about the origin. If a and b are two points on Γ_L, not antipodes of one another, then we can orient Γ_L from a to b along the circle route of $< 180°$.

If A and B are oriented lines in OP^3, multiplying the coordinates of A and B by positive scalars will not alter the sign of $A \vee B$. We define the *linking number* of two oriented lines, denoted $\mathbf{link}\,(A, B)$, by the expression

$$\mathbf{link}\,(A, B) \;=\; \mathbf{sign}\,(A \vee B).$$

Since $A \vee B$ can be expressed as a bracket, it is a projective invariant. It follows that $\mathbf{link}\,(A, B)$ is *preserved* by projective transformations of positive determinant, such as rotation or translation of $\mathcal{R}^3$, *altered* by projective transformations of negative determinant, such as reflection in a point in $\mathcal{R}^3$. Furthermore, since $A \vee B$ is zero if and only if A and B are coplanar, $\mathbf{link}\,(A, B)$ is an invariant for the isotopy of oriented lines. Since lines are of even rank, the join operation $\vee$ is commutative, and

$$\mathbf{link}\,(A, B) = \mathbf{link}\,(B, A), \quad \text{while}$$
$$\mathbf{link}\,(-A, B) = \mathbf{link}\,(A, -B) = -\mathbf{link}\,(A, B).$$

Recall also that

$$\chi_{ABC} \;=\; \mathbf{link}\,(A, B) \times \mathbf{link}\,(A, C) \times \mathbf{link}\,(B, C),$$

independent of orientation. The next proposition provides an easy visualization of the linking number of two skew oriented lines A and B, using the visualizations of the given oriented lines. We here assume that the Euclidean space $\mathcal{R}^3$ is provided with a "positive" reference frame (right-hand rule), and is given the standard embedding in OP^3.

Proposition 8. *Oriented skew lines have positive linking number if and only if their orientations follow the "left-hand rule", that is, if the thumb of the left hand indicates to the orientation of one line, the fingers wrap around it in the direction given by the orientation of the other line.*

Proof: Consider a pair of oriented lines, neither of which lies in the plane at infinity. By letting them approach one another, without letting them meet, it will be seen that their orientation is given either by the "right hand" or by the "left hand", not both. Any two positions of the left hand are obtainable, one from the other, by an isotopy. Thus it suffices to show that one position of the left hand has positive linking number. Try the positive x-axis for the thumb, the line parallel to the positive y-axis at $z = 1$, for the fingers. Then

$$\mathrm{link}\,(T, F) \;=\; \mathrm{sign}\,((0,0,0,1) \vee (1,0,0,0) \vee (0,0,1,1) \vee (0,1,0,0) \;=\; 1$$

If one of the lines lies on the plane at infinity, the same principle applies. Taking the thumb of the left hand as before, along the positive x-axis, the fingers indicating the rotation from the positive z axis toward the positive y-axis, we replace $(0,0,1,1)$ by $(0,0,1,0)$ in the above formula, without changing the value of the bracket. □

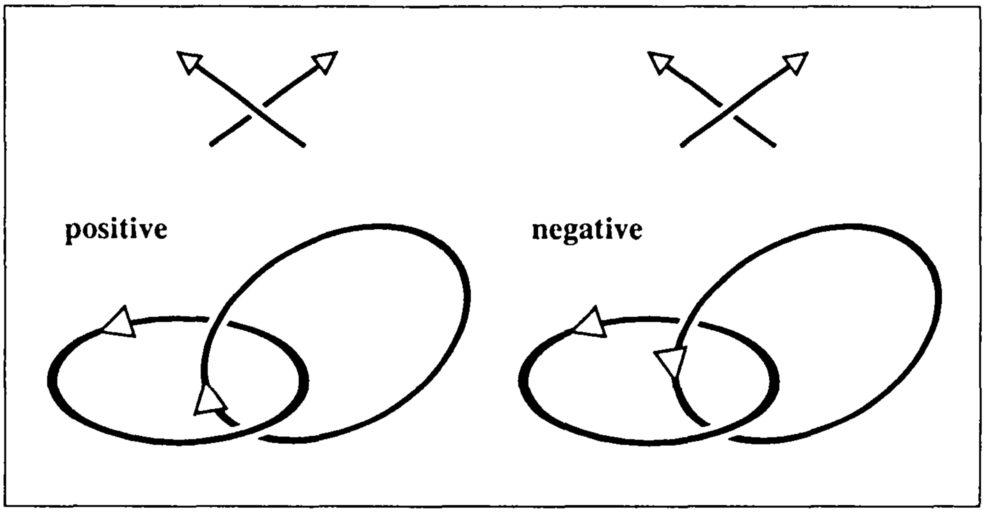


Figure 5. Comparison of linking numbers, for oriented lines and for oriented links.

The term "linking number" comes from the knot theory. We make the connection clear in Sections 10,11 and 12. Sufficeth to say for the moment that positive and negative linkings of unknotted, oriented circular loops are as shown in Figure 5. Observe that wherever we interpret short segments of these loops as oriented lines, their linking number is opposite to that indicated for the given loop.[7]

Fix an oriented point z in $\mathcal{P}^3$ as a *view point* or *center of perspective* from which to view the entire projective space. For oriented points a, b distinct from and collinear with z, we say that

$$\begin{aligned} z \textit{ unites } a \textit{ with } b \quad &\Leftrightarrow \quad \omega(z \vee a,\; z \vee b) \;=\; +1. \\ z \textit{ separates } a \textit{ from } b \quad &\Leftrightarrow \quad \omega(z \vee a,\; z \vee b) \;=\; -1. \end{aligned}$$

In particular, any oriented point $z \neq \pm a$ separates a from its negative, $-a$.

To each pair A, B of oriented skew lines not through z there corresponds a line

$$\mathrm{turn}_z(A, B) \;=\; (z \vee A) \wedge (z \vee B)$$

through z, meeting both A and B. We call this oriented line the *turn from A to B, as seen from the point z*. The corresponding unoriented line K is called the *directrix* through z, meeting A and B. Let k_A, k_B be the oriented points

$$\begin{aligned} k_A &= (z \vee B) \wedge A \\ k_B &= (z \vee A) \wedge B \end{aligned}$$

[7] The correspondence between lines and circles, via the Klein map, will be developed in Section 10.

where the directrix through z meets the two lines. If A and B are skew lines then the join of these points is a scalar multiple of the directrix, which we call

$$\mathbf{after}\ _z(A, B) \quad = \quad k_A \vee k_B.$$

Since **turn** and **after** are equal as projective lines, we may compare their orientations.

Theorem 9. *For skew oriented lines A, B not incident with a point z,*

$$\omega(\mathbf{turn}\ _z(A, B),\ \mathbf{after}\ _z(A, B)) \quad = \quad \mathbf{link}\ (A, B).$$

Proof: *Let $a = k_A, b = k_B$, and let a', b' be other points on the lines A, B, respectively, such that $a \vee a' = A$, $b \vee b' = B$. Then*

$$\begin{array}{rclclcl} \mathbf{turn}\ _z(A,B) & = & zaa' \wedge zbb' & = & [a'zbb']za & = & [zbb'a']az \\ k_A & = & (z \vee B) \wedge A & = & zbb' \wedge aa' & = & a[zbb'a'] \\ k_B & = & (z \vee A) \wedge B & = & zaa' \wedge bb' & = & -[za'bb']a \ + \ [aa'bb']z, \end{array}$$

so

$$\mathbf{after}\ _z(A, B) \quad = \quad [zbb'a'][aa'bb']az$$

and

$$\omega(\mathbf{turn}\ _z(A, B),\ \mathbf{after}\ _z(A, B)) \quad = \quad \mathbf{sign}\ ([aa'bb']) \quad = \quad \mathbf{link}\ (A, B)$$

□

It should be noted that both **turn** $_z(A, B)$ and **after** $_z(A, B)$ depend on the position of z, while **link** (A, B) does not. On the other hand, both **turn** $_z(A, B)$ and **link** (A, B) depend on the orientations of the lines A and B, while **after** $_z(A, B)$ does not. Recall that the linking numbers are isotopy invariants of oriented lines. From Theorem 9 it follows that the relative orientation of **turn** $_z(A, B)$ and **after** $_z(A, B)$ also remains invariant under an isotopy during which they remain defined, that is, during which no line meets the center z. Such an isotopy of lines (during which no line passes through the center of perspective z) is called a *perspective isotopy.* We say that **turn** and **after** are *concomitants of perspective isotopy.*

Theorem 10. *Let A, B be oriented skew lines not passing through a point z, such that the directrix K through z meets A and B at finite points. Let d_A, d_B be the directions of these lines. Then*

$$\mathbf{link}\ (\mathbf{turn}\ _z(A, B),\ d_A \vee d_B) \quad = \quad \omega(z \vee a,\ \ z \vee b),$$

where a, b are positive points where K meets A and B, respectively. The link is thus positive if and only if the point z unites (does not separate) the points a and b.

Proof: Let a, b be such positive points on the directrix $K = \mathbf{turn}\ _z(A, B)$. Then $A = a \vee d_A$, $B = b \vee d_B$. So the indicated link is given by the expression

$$\begin{array}{cl} & (zad_A \wedge zbd_B)d_Ad_B \\ = & [zzbd_B][ad_Ad_Ad_B] - [azbd_B][zd_Ad_Ad_B] + [d_Azbd_B][zad_Ad_B] \\ = & [zad_Ad_B][zbd_Ad_B] \\ = & \omega(z \vee a,\ \ z \vee b) \end{array}$$

□

Corollary 11. *If oriented lines A and B are viewed from a center of perspective z along the ray $K = \mathbf{turn}\ _z(A, B)$, and if K meets A and B at positive points which are united (not separated) by z,* *then the oriented line A, viewed along K, requires a CCW turn of less than 180° to match the orientation of line B.*

Proof: Using the notation of Theorem 10, if a and b are united by z, then

$$\omega(z \vee a, \;\; z \vee b) \;\; = \;\; +1,$$

so the **link** of **turn** $_z(A, B)$ with the line at infinity $d_A \vee d_B$ is positive, and is given by the left-hand rule. So the rotation from the direction of A to the direction of B, seen along **turn** $_z(A, B)$ is CCW. □

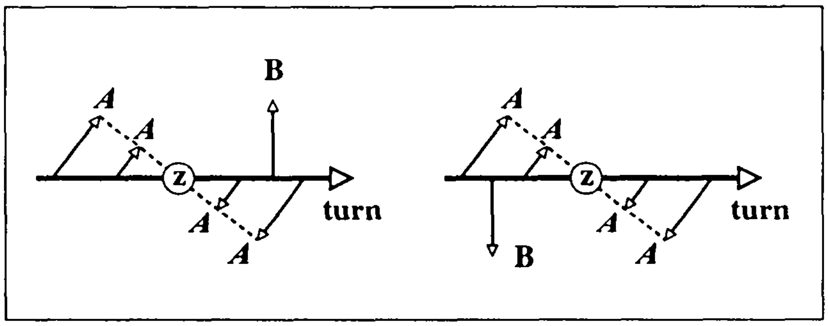


Figure 6. Pairs of oriented lines A, B with equal **turn** relative to a center z.

Observe that **turn** $_z(A, B)$ depends only upon the oriented planes $z \vee A$, $z \vee B$, so a modification of lines which does not alter the orientations of these planes will not change the orientation of **turn** $_z(A, B)$. This permits an extension of Corollary 11 which can be used for any pair of lines A, B. Indeed, we can go so far as to project the lines A and B from z into a single plane. The resulting lines A' and B', oriented in such a way that $z \vee A = z \vee A'$ and $z \vee B = z \vee B'$ as oriented planes, satisfy

$$\textbf{turn}\;_z(A, B) \;\; = \;\; \textbf{turn}\;_z(A', B').$$

Figure 6 illustrates this phenomenon. For all the indicated positions of the lines A, B, the directrix **turn** $_z(A, B)$ is oriented to the ***right*** on this page, as shown. When the center unites the points where these lines meet the directrix, the orientation of **turn** $_z(A, B)$ is given by the left-hand rule relative to directions of the lines A, B. Equivalently, the projections from z of the oriented lines A, B onto a fixed, but arbitrary, plane do not change (see Section 5, below).

Theorem 12. *Let A, B be oriented skew lines not passing through a point z, such that the directrix K through z meets A and B at finite points. Let*

$$s_A \;\; = \;\; \textbf{sign}\,(k_A), \;\; s_B \;\; = \;\; \textbf{sign}\,(k_B),$$

be the signs of the points k_A, k_B where the directrix meets the lines A, B, respectively. If a and b denote the positive orientations of k_A and k_B, respectively, then

$$s_A s_B \;\; = \;\; -\omega(za, zb),$$

a quantity which is positive if and only if the center z of perspective separates the points a, b. Furthermore,

$$\textbf{relOrient}\,(\textbf{after}\;_z(A, B), (z, b, a)) \;\; = \;\; +1.$$

Proof: Since

$$k_A = zbd_B \wedge ad_A = a[zbd_Bd_A],$$

the sign s_A is that of the bracket $[zbd_Bd_A]$. Similarly for s_B. We have

$$\begin{aligned} s_A &= [zbd_Bd_A] = -[zbd_Ad_B] \\ s_B &= [zad_Ad_B] \end{aligned}$$

so

$$s_As_B = -\omega(za, zb)$$

The relative orientation of $K = \text{after}_z(A, B)$ with respect to the triple (z, b, a) of points is given by the product of signs

$$\omega(K, zb)\ \omega(K, za)\ \omega(K, ba).$$

Since $k_A = s_Aa$, $k_B = s_Bb$, the relative orientation can be computed as

$$\begin{aligned} &\omega(s_As_Bab, zb)\ \omega(s_As_Bab, za)\ \omega(s_As_Bab, ba) \\ &= s_As_B\omega(ab, zb)\ \omega(ab, za)\ \omega(ab, ba) \\ &= -s_As_B\omega(ab, zb)\ \omega(ab, za) \\ &= -s_As_B\omega(za, zb) \end{aligned}$$

a product which is equal to $+1$, by the above calculation of s_As_B. □

Since, by Theorem 12, the line after has the orientation of the triple (zba), we find justification for our choice of name for this concomitant.

In what follows, we shall often concentrate on one particularly simple case of the calculation of the perspective concomitant after. Taking the center of perspective at $(0, 0, 1, 0)$ the point at infinity on the positive z-axis, the sign $\text{after}_z(A, B)$ is positive for lines A, B if and only if the line A would naturally appear to pass *under* the line B when projected into the xy-plane. In Section 5, we will use this idea in order to define a sign $\text{under}(A', B')$ for lines A', B' which cross in a plane of projection.

5. Projections of oriented lines

A *projection* $\pi = \pi_{z,Q}$ is determined by a point z, the *center* of projection, together with a plane Q, the *screen*, not containing the point z. The image p' of any point p under a projection is given by the expression

$$p' = \pi_{z,Q}(p) = (z \vee p) \wedge Q.$$

A projection $\pi_{z,Q}$ is said to be ***positive*** if and only if the bracket $z \vee Q = [zQ]$ is a positive number (pseudoscalar). A ***ray*** of a projection $\pi_{z,Q}$ is any oriented line through z. We say a ray K is directed ***toward*** an oriented point q on the screen Q if and only if

$$\omega(z \vee q, K) = +1.$$

Also, we say the ray K is ***directed toward the screen*** Q if and only if K is directed toward a positive point on Q, otherwise we say that K is directed *away* from the screen. But when K is parallel to the screen, "toward" and "away" are no longer appropriate: we can only say that K is directed toward the point d_K, its direction, on Q.

Proposition 13. *For any projection $\pi_{z,Q}$, for any oriented point $p \neq z$, and for any oriented point q in the plane Q and collinear with z and p,*

$$\pi_{z,Q}(p) \sim \omega(zp, zq)\ [zQ]\ q.$$

Thus, under a positive projection $\pi = \pi_{z,Q}$, the image $\pi(p)$ of a positive point is positive if and only if the point z unites (does not separate) the point p and its image on the screen. Under these conditions, the ray $z \vee p$ is directed toward the screen.

Proof:

$$\pi_{z,Q}(p) = (z \vee p) \wedge Q \sim \omega(zp, zq)\ zq \wedge Q = \omega(zp, zq)\ [zQ]\ q,$$

by Theorem 1 of Section 2. Let q be the positive point on Q and on $z \vee p$, as above. Then $\pi(p) \sim q$ if and only if $\sigma(zp, zq) = +1$, if and only if z unites the points p, q. □

As a special case of Proposition 13, consider any two distinct finite points p, q in $\mathcal{P}^3$. The line $L = p \wedge q$ has direction

$$d_L = \pi_{p,I}(q)$$

given by the projection of q from center p onto the plane I at infinity.

Proposition 14. *The image $a' \vee b'$ of a line $L = a \vee b$ not passing through the point z, under a projection $\pi_{z,Q}$, is given by the expression*

$$L' = \pi_{z,Q}(L) = [zQ]\ (z \vee L) \wedge Q.$$

If $\pi_{z,Q}$ is positive, $a' \vee b'$ and $(z \vee L) \wedge Q$ are the same oriented line.

Proof:

$$\begin{aligned}(za \wedge Q) \vee (zb \wedge Q) &= ([zQ]a - [aQ]z) \vee ([zQ]b - [bQ]z) \\ &= [zQ]\ ([zQ]ab - [aQ]zb - [bQ]az) \\ &= [zQ]\ (zab \wedge Q) = [zQ]\ (z \vee L) \wedge Q.\end{aligned}$$

□

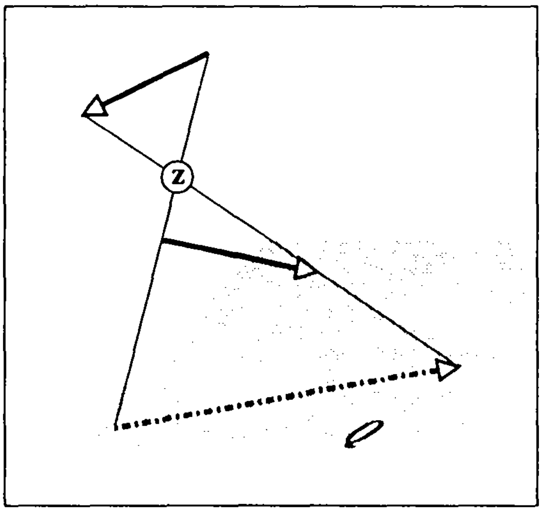


Figure 7. Projections of perspective-isotopic lines.

Corollary 15. *Let $\pi_{z,Q}$ be a fixed projection. Let $\{L(t); 0 \leq t \leq 1\}$ be a perspective isotopy of an oriented line L relative to the center of perspective z. Assume further that the lines $L(t)$ all lie in the initial plane $z \vee L$. Then the projected image $L'(t) = \pi_{z,Q}(L(t))$ is constant as an oriented line, with value $L' = \pi_{z,Q}(L)$.*

Proof: This is a consequence of Proposition 14, since $L'(t)$ is a well-defined (non-zero) oriented line at any time t, having a fixed underlying projective line, whose coordinates vary continuously with t. □

We are here simply underlining the fact that for a fixed projection $\pi = \pi_{z,Q}$, the image $\pi(L) = (z \vee L) \wedge Q$ of a line L depends only on the orientation of the plane $z \vee L$. Distinct oriented lines which give this plane the same orientation will have the *same* orientation in projection. See Figure 7.

In Section 7 we will define the *planar layout* of a line configuration. A planar layout is the projection of the configuration into a plane, together with certain "depth" or "crossing" data. To this end, we now settle some matters concerning sign functions relative to a projection, useful for retaining information about the relative position of a pair of oriented lines.

Recall that the sign of an oriented point is defined by the sign of its fourth coordinate. Let $\pi_{z,Q}$ be a fixed projection. Let K be a ray through the center z, and let q be the positive point of Q that lies on the ray K. Then we can define the sign

$$\textsf{toward}\,(K) = \omega(z \vee q, K),$$

which is positive when the ray K is directed from the center z toward the screen Q across a finite region of $\mathcal{P}^3$.

A projection $\pi_{z,Q}$ and a configuration L are called *proper* with respect each other if the point z meets no line of L and if every directrix K through z meeting a pair of lines of L meets Q in a finite point. If L is proper with respect to $\pi_{z,Q}$ then L projects to a family of n distinct lines in Q, none of which are at infinity, and no pair of which are parallel. Using such a projection, we can study the projected configuration $\pi_{z,Q}(L)$ as a figure in an affine plane, the set of finite points of Q. Assume that the projection is positive, with $[zQ] = 1$, so that the projection of oriented lines (see Proposition 14) is given by

$$L' = \pi(L) = (z \vee L) \wedge Q.$$

If the projection $\pi_{z,Q}$ is proper with respect to the configuration L we can define, for the images

$$A' = \pi(A),\ B' = \pi(B)$$

of each pair A, B of oriented lines in L, the following sign functions

$$\begin{aligned} \textsf{link}\,(A', B') &= \textsf{link}\,(A, B) \\ \textsf{turn}\,(A', B') &= \textsf{toward}\,(\textsf{turn}_z(A, B)) \\ \textsf{under}\,(A', B') &= \textsf{toward}\,(\textsf{after}_z(A, B)). \end{aligned}$$

Theorem 16. *Let L' be the image of a configuration L under a proper projection $\pi_{z,Q}$. Then*

$$\textsf{link}\,(A', B') = \textsf{turn}\,(A', B') \times \textsf{under}\,(A', B').$$

Proof: By Theorem 9 of Section 4, we have

$$\begin{aligned} \textsf{link}\,(A', B') &= \textsf{link}\,(A, B) \\ &= \omega(\textsf{turn}_z(A, B), \textsf{after}_z(A, B)) \\ &= \textsf{toward}\,(\textsf{turn}_z(A, B)) \times \textsf{toward}\,(\textsf{after}_z(A, B)) \\ &= \textsf{turn}\,(A', B') \times \textsf{under}\,(A', B'). \end{aligned}$$

□

Corollary 17. *Let A and B be two oriented skew lines in $\mathcal{R}^3$, equipped with a positive reference frame. Let A' and B' be their respective projections, using a proper positive projection (central or parallel). Then* $\textsf{link}\,(A,B)=+1$ *if and only if the "undercrossing" line, viewed from the center of projection toward the screen, requires a CCW turn of less than 180° to match the orientation of the other line.*

Proof: Suppose that A' is the "undercrossing" line, that is,

$$\textsf{under}\,(A',B')=+1.$$

From Corollary 11 it follows that A' needs a CCW turn to match the orientation of B' if and only if $\textsf{turn}_{\,z}(A,B)$ is oriented toward the screen of projection, which on its turn is equivalent to

$$\textsf{turn}\,(A',B') \;=\; \textsf{toward}\,(\textsf{turn}_{\,z}(A,B)) \;=\; +1,$$

and hence to

$$\textsf{link}\,(A,B) \;=\; \textsf{link}\,(A',B') \;=\; +1$$

by Theorem 16. □

Next, consider a continuous path

$$H\;:\;[0,1]\to(G_{2,4})^n,$$

(not necessarily in C, that is, possibly allowing pairs of lines to meet). Let H such that $\pi_{z,Q}$ is a proper projection with respect to each $H(t)$. This means that for each t, if I denotes the line at infinity of Q,

(1) no line of $H(t)$ contains z,

(2) no pair of lines of $H(t)$ generates together with I a ruled surface that contains z.

In this case we say that the path H is *proper* with respect to $\pi_{z,Q}$.

Theorem 18. *If H is a continuous path in $(G_{2,4})^n$ that is proper with respect to the given projection $\pi_{z,Q}$, connecting the sets of lines $L=\{L_1,\ldots,L_n\}$ at $t=0$ to $\mathcal{M}=\{M_1,\ldots,M_n\}$ at $t=1$, and if all lines are oriented such that each M_i inherits its orientation from L_i along $H(t)$ then*

$$\textsf{turn}\,(L_i',L_j')=\textsf{turn}\,(M_i',M_j')$$

for each pair $\{i,j\}\subset\{1,\ldots,n\}$.

Proof: If q is the positive point corresponding to $A'\cap B'$ then

$$\textsf{turn}\,(A',B')=\omega(z\vee q,\textsf{turn}_{\,z}(A,B)).$$

The coordinates of $z\vee q$ and of $\textsf{turn}_{\,z}(A,B)$ are continuous functions (polynomials) in the coordinates of A and B. By the assumptions on the path $h(t)$, the coordinates of A and B vary continuously with t, such that at any time t the lines $z\vee q$ and $\textsf{turn}_{\,z}(A,B)$ are well-defined, whence the assertion follows. □

Theorem 19. *If $L(t)$ is an isotopy of line configurations, for $0\le t\le 1$, connecting $L(0)=L$ to $L(1)=\mathcal{M}$, such that $L(t)$ is proper with respect to a given projection $\pi_{z,Q}$, then*

$$\textsf{under}\,(L_i',L_j')=\textsf{under}\,(M_i',M_j')$$

for each pair $\{i,j\} \subset \{1,\ldots,n\}$.

Proof: Recall that under (A', B') does not depend on the chosen orientation of A and B, and that

$$\text{under}\,(A', B') = \text{link}\,(A', B') \times \text{turn}\,(A', B').$$

We give L arbitrary orientations, and let $\mathcal{M}$ inherit its orientations from it by $L(t)$. The assertion follows from the invariance of link for an isotopy of oriented configurations, and from the invariance of turn for a path that is proper with respect to the projection (Theorem 18). □

6. Limit-isotopies of line configurations

The passage from a configuration of skew lines in $\mathcal{P}^3$ to line diagrams in the plane with "crossing information" is a special case of a general construction, which we shall call "projective limit". We introduce an appropriate notion of proximity directly in the projective space.

Definition. A parametrized set $\{A_t\}$ of coordinates

$$A_t \;=\; (A_I(t);\; I \in \binom{N}{k})$$

for projective flats of rank k in $\mathcal{P}^n$, for t in a punctured open neighborhood of s in the one-point compactification $\mathcal{R}^\bullet \;=\; \mathcal{R} \cup \infty$ of the reals, has as *projective limit* a flat

$$\lim_{t \to s} A_t \;=\; B \;=\; (B_I;\; I \in \binom{N}{k})$$

as $t \to s$, if and only if there exists a real-valued function $\rho : \mathcal{R} \to \mathcal{R}$ such that in $\mathcal{R}^{n+1}$,

$$lim_{t \to s}\; \rho(t) A_I(t) \;=\; B_I, \;\text{ for all }\; I \in \binom{N}{k}.$$

For example, the projective limit as $t \to 0$ of the *projectively constant* function tp, for a scalar t and projective point p is (up to a *non-zero* scalar multiple)

$$lim_{t \to 0}\, tp \;=\; p$$

rather than the zero vector $(0,0,0,0)$, as it would be if the limit were taken in $\mathcal{R}^4$.

We now use the notion of *limit-isotopy* to "move" all of $\mathcal{P}^3$ to its image under a projection $\pi_{z,Q}$. For any projection $\pi_{z,Q}$ of $\mathcal{P}^3$, the parametrized family of projective maps $\sigma_t \;=\; \sigma_{t,z,Q}$ given by the formula

$$\sigma_t(p) \;=\; t(z \vee Q)p + (1-t)(p \vee Q)z, \;\text{ for }\; 0 < t < \infty \tag{9}$$

is called the *limit isotopy of the projection* $\pi_{z,Q}$.

Theorem 20. *For any projection* $\pi_{z,Q}$ *of* $\mathcal{P}^3$, *the parametrized family of projective maps* $\sigma_t \;=\; \sigma_{t,z,Q}$ *given by the formula*

$$\sigma_t(p) \;=\; t(z \vee Q)p + (1-t)(p \vee Q)z, \;\textit{ for }\; 0 < t < +\infty$$

is a linear isotopy of $\mathcal{P}^3$ *with the properties (limits are taken in* $\mathcal{P}^3$*):*

$$\lim_{t\to 0}\ \sigma_t(p) \quad = \quad \begin{cases} z, & \text{for all points } p \text{ not on the plane } Q, \\ p & \text{for points } p \text{ on } Q. \end{cases}$$

$$\sigma_1(p) \quad = \quad \begin{cases} p \text{ for all points } p \in \mathcal{P}^3. \end{cases}$$

$$\lim_{t\to\infty}\ \sigma_t(p) \quad = \quad \begin{cases} \pi_{z,Q}(p) & \text{for all points } p \neq z, \\ z, & \text{for } p = z. \end{cases}$$

Proof: For any fixed t in the range $0 < t < +\infty$, and for any line K through the center z, $\sigma_t(K) = K$, each point p being mapped to a linear combination of z and p. It follows that each map σ_t is a non-singular map, and that σ_t; $0 < t < +\infty$ is an isotopy.

Writing the flats of rank 4 as brackets, we have the equivalent expressions

$$\sigma_t(p) \;=\; t[zQ]p + (1-t)[pQ]z \;\sim\; [zQ]p + (\frac{1}{t} - 1)[pQ]z$$

For points p on the plane Q, $\sigma_t(p) = t[zQ]p$ is a non-zero real multiple of the point p, for all $t \neq 0$. So $\lim_{t\to s} \sigma_t(p) = p$ for all $s \in [0, +\infty]$.

For $p = z$, $\sigma_t(p) = [zQ]z$ is a non-zero real multiple of the point z, so $\lim_{t\to s} \sigma_t(z) = z$ for all $s \in [0, +\infty]$.

For points p not equal to z and not lying in the plane Q,

$$\lim_{t\to 0} \sigma_t(p) \;=\; [pQ]z \;\sim\; z,$$

while

$$\begin{aligned} \lim_{t\to\infty}\ \sigma_t(p) &= \lim_{t\to\infty} ([zQ]p + (\tfrac{1}{t} - 1)[pQ]z \\ &= [zQ]p - [pQ]z \;=\; (z \vee p) \wedge Q \;=\; \pi_{z,Q}(p). \end{aligned}$$

□

Theorem 21. *The action of the isotopy* σ_t *on a line* L *in* $\mathcal{P}^3$ *is given by the expression*

$$\sigma_{t,z,Q}(L) \;=\; t[zQ]L \;+\; (1-t)(z \vee (L \wedge Q))$$

with limits

$$\lim_{t\to 0}\ \sigma_t(L) \quad = \quad \begin{cases} z \vee (L \wedge Q) & \text{for all lines } L \text{ not lying in the plane } Q, \\ L & \text{for lines on } Q. \end{cases}$$

$$\lim_{t\to 1}\ \sigma_t(L) \quad = \quad \begin{cases} L \text{ for all lines } L \in \mathcal{P}^3. \end{cases}$$

$$\lim_{t\to\infty}\ \sigma_t(L) \quad = \quad \begin{cases} \pi_{z,Q}(L) & \text{for all lines } L \text{ not incident with } z, \\ L, & \text{for lines } L \text{ incident with } z. \end{cases}$$

This action $\sigma_{t,z,Q}$ *is an isotopy of projective lines.*

Proof: When it is applied to a line $L = a \vee b$, we have

$$\begin{aligned} &\sigma_{t,z,Q}(L) \\ &= (t[zQ]a + (1-t)[aQ]z) \vee (t[zQ]b + (1-t)[bQ]z) \\ &= t^2[zQ]^2 L \;+\; t(1-t)[aQ][zQ]zb \;+\; t(1-t)[zQ][bQ]az \\ &\sim t[zQ]L \;+\; (1-t)([aQ]zb + [bQ]az) \\ &= t[zQ]L \;+\; (1-t)(z \vee (L \wedge Q)) \end{aligned}$$

For any line L lying in the plane Q, $L \wedge Q = 0$, so $\sigma_t(L) = t[zQ]L$ is a real multiple of L, for all $t \neq 0$. Thus $\lim_{t\to s}(L) = L$ for all $s \in [0, +\infty]$

For any line L passing through the point z, $z \vee (L \wedge Q) = [zQ]L$, a multiple of the line L.

For lines L not on Q,

$$\lim_{t\to 0} \sigma_t(L) = z \vee (L \wedge Q).$$

For any line L,

$$\lim_{t\to 1} \sigma_t(L) = t[zQ]L \sim L.$$

For lines L not incident with z,

$$\begin{aligned}\lim_{t\to\infty} \sigma_t(L) &= \lim_{t\to\infty} [zQ]L + (\tfrac{1}{t} - 1)(z \vee (L \wedge Q)) \\ &= (z \vee Q) \wedge L - z \vee (L \wedge Q) \\ &= (z \vee L) \wedge Q = \pi_{z,Q}(L),\end{aligned}$$

by Theorem 1.

Because $(\sigma_t)_{t>0}$ is defined on projective points, and is an ambient isotopy of $\mathcal{P}^3$, disjoint lines $A = \sigma_1(A)$ and $B = \sigma_1(B)$ remain disjoint at any time $t > 0$, so the action of (σ_t) is an isotopy of lines. □

Let L be a *degenerate* configuration in which *every* pair of lines intersects. There are two possibilities for such a configuration. L is either a *planar configuration*, with all its lines in one plane, or it is a 3-*pencil*, with all its lines passing through one point in $\mathcal{P}^3$. Theorem 21 shows that any given configuration can be moved, by means of an isotopy, arbitrarily close to completely degenerate configurations of these two types. This is accomplished by letting t approach 0 or $+\infty$, in the parametrization of $\sigma_{z,Q}$. This reasoning leads us to the conclusion that every cell (connected component) of $\mathcal{C}$ has these two types of degeneracies in its boundary (with respect to the locally euclidean topology of $(G_{2,4})^n$). It turns out that, under this limit-isotopy, a given line configuration approaches the degenerate limit configurations in a very controlled fashion.

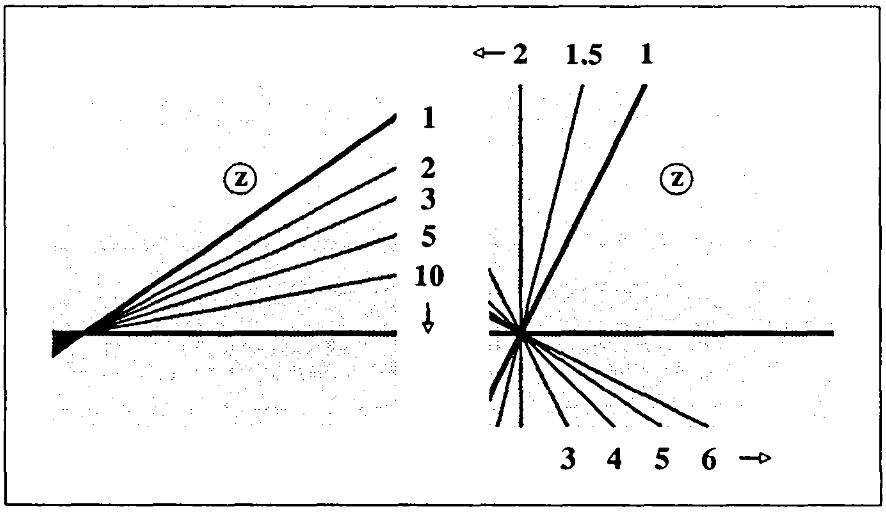


Figure 8. The action on lines of the limit isotopy of a central projection.

Theorem 22. *For any line A not through z, its image A_t during the isotopy $\sigma_{z,Q}$ never meets the point z, for $0 < t < \infty$. Furthermore, its family of images L_t is the pencil of lines through the point $L \wedge Q$ and in the plane $z \vee L$.*

Proof: The join

$$\begin{aligned} z \vee A_t &= z \vee (t[zQ]A + (1-t)(z \vee (A \wedge Q)) \\ &= t[zQ](z \vee A), \end{aligned}$$

is a non-zero multiple of the fixed plane $z \vee A$, for $0 < t < \infty$.

All lines $\sigma_t(L)$ are linear combinations of the lines $[zQ]L$ and $z \vee (L \wedge Q)$, both of which are in the flat pencil through $L \wedge Q$ in the plane $z \vee L$, and are therefore also in that pencil. □

Since it is often convenient to know for what value of t a point $\pi_{z,Q}(p)$ passes through the plane at infinity, we carry out the following calculation.

Theorem 23. *For any point p separated by z from its image $p' = \pi_{z,Q}$, its image $\sigma_t(p)$ will reach the plane at infinity at one positive (finite) value of t. Otherwise, there is no such value of t in that range.*

Proof: The fourth coordinate of $\sigma_t(p)$ is given by the expression

$$t[zQ]p_4 \;+\; (1-t)[pQ]z_4$$

which is zero when

$$\frac{1}{t} \;=\; 1 - \frac{[zQ]p_4}{[pQ]z_4}.$$

This one value of t is positive ($0 < t < \infty$), and is thus in the range of the parameter for the isotopy, if and only if

$$0 \;<\; [pQ]z_4 \;-\; [zQ]p_4 \;=\; -(zp \wedge Q)_4,$$

since $zp \wedge Q \;=\; [zQ]p - [pQ]z$. But this is the condition that the point $p' = \pi_{z,Q}(p)$ is negative, or equivalently, by Proposition 13, that z separates p and p'. □

Figure 8 shows the action of the limit isotopy of the projection $\pi_{z,Q}$ where $z = (0,0,1,1)$ and Q is the xy-plane, as seen from a point on the negative y-axis. Lines in the xz-plane meeting the z-axis at points *beneath* the center $(0,0,1,1)$ rotate, about their intersection with the x-axis, arriving at that axis in the limit *from above*. Lines meeting the z-axis at points $(0,0,h,1)$ *above* the center rotate in the opposite direction, pass through a *vertical* position at $t = -h/(1-h)$, and approach the x-axis *from below*. For all lines L not passing through the center z, an analogous action takes place in the plane $z \vee L$.

Corollary 24. *Let L be a proper configuration with respect to the projection $\pi_{z,Q}$, and let $\sigma = \sigma_{z,Q}$ denote the limit-isotopy. For each pair $\{A, B\}$ of L we put A'_t and B'_t be the projections of $\sigma_t(A)$ and $\sigma_t(B)$, respectively. Then* $\text{under}\,(A'_t, B'_t)$ *is constant for $0 < t < +\infty$ and* $\text{turn}\,(A'_t, B'_t)$ *is constant for $0 < t \leq +\infty$.*

Proof: By Theorem 22 the whole family $\{L_t;\ 0 < t \leq +\infty\}$ is proper with respect to the projection $\pi_{z,Q}$. We conclude, by Theorem 18 of Section 5, that the signs $\text{turn}\,(A'_t, B'_t)$ do not change. Furthermore, since $\{\sigma_t; 0 < t < +\infty\}$ is an isotopy, the signs $\text{under}\,(A'_t, B'_t)$ are constant, due to Theorem 19. For parameter values $0 < t \leq +\infty$, we have $A'_t \sim A'$ (equality as oriented lines) by Corollary 15. □

One application of the limit-isotopy constructed in this section involves the *affine isotopy problem*. Suppose we are given two *affine configurations*, that is, two labelled sets of n pairwise disjoint lines in

$\mathcal{R}^3$. Recall that in an affine configuration, lines can be parallel. Under an *affine isotopy* no line moves (vanishes) into the plane at infinity, and each pair of lines remains affinely disjoint (have no finite point of intersection). Can any two affine configurations be connected by an affine isotopy? They answer turns out to be "yes" [Vi].

Theorem 25. *Any two affine configurations are affinely isotopic.*

Proof: We show that any affine configuration L is affinely isotopic to a pencil of parallel lines. Since any two such pencils of n lines are affinely isotopic, the theorem will follow.

If L is a given affine configuration then we can always choose a transverse plane Q, such that Q does not contain any line of L, then choose a point z at infinity, not on Q. The limit isotopy $\sigma_{z,Q,t}(L)$, for $1 \geq t \geq 0$, connects L at $t = 1$ with a pencil through z at $t = 0$.

We must check that no line vanishes, and that parallel lines (projectively concurrent) do not become affinely concurrent during the limit isotopy. Let p be another point at infinity, so $p \vee z$ is a line at infinity. Because $\sigma_t(p)$ is collinear with p and z for every $t > 0$, by the definition of σ_t, $\sigma_t(p)$ remains at infinity during the whole isotopy. If two lines of L intersect, they do so at an infinite point, and hence remain affinely disjoint in $\sigma_t(L)$ for every $t \in [0, 1]$. By the same argument, no line vanishes during the isotopy.

We conclude that $\{\sigma_t; 1 \geq t \geq 0\}$ is an affine isotopy for L, bringing all lines parallel to each other, in the pencil through z. □

7. Planar layouts

Configurations of lines in 3-space are usually "sketched" as arrangements of lines in the plane, with additional "crossing" information. The crossings are indicated in such a sketch by "lifting the pen"' whenever a line is supposed to pass "under" another line. The material in this section will permit us to make these considerations precise.

Extend the affine plane to form a closed disc A by adding the circle of semiprojective points at infinity. An (oriented) point at infinity and its negative are said to be *antipodal*. A *pseudoline arrangement* is any finite set of non-self-intersecting curves in A, joining distinct pairs of antipodal points at infinity, such that each two curves intersect each other in exactly one point, where they transversally cross. Such an arrangement is *stretchable* if it is equivalent, under an orientation-preserving homeomorphism of the open disc (the affine plane), to an arrangement of straight lines. A *pseudoline diagram* H is a pseudoline arrangement together with an antisymmetric function under , assigning a sign

$$\begin{aligned} \textbf{under}\,(A,B) &= \pm 1 \\ \textbf{under}\,(B,A) &= -\textbf{under}\,(A,B) \end{aligned}$$

to each ordered pair A, B of distinct lines. To each pseudoline diagram we associate a *drawing*, in which the sign under $(A, B) = +1$ is indicated by omitting a small portion of the drawing of pseudoline A in a neighborhood of its intersection with pseudoline B. See Figure 9. A *line diagram* $\mathcal{D}$ is a pseudoline diagram realizable in the affine plane using straight lines.

Given any configuration L of lines in $\mathcal{P}^3$, and any projection $\pi_{z,Q}$ onto the plane Q, for each line $A \in L$ let A' denote its image $A' = \pi_{z,Q}(A)$. Recall from Section 6 that the planar arrangement

$$L' = \{A' ; A \in L\}$$

is the limit of the configuration L under the isotopy $\sigma_{z,Q}$. If, moreover, the configuration is proper with respect to the given projection, then we may define

$$\textbf{under}\,(A', B') = \textbf{toward}\,(\textbf{after}_z(A, B))$$

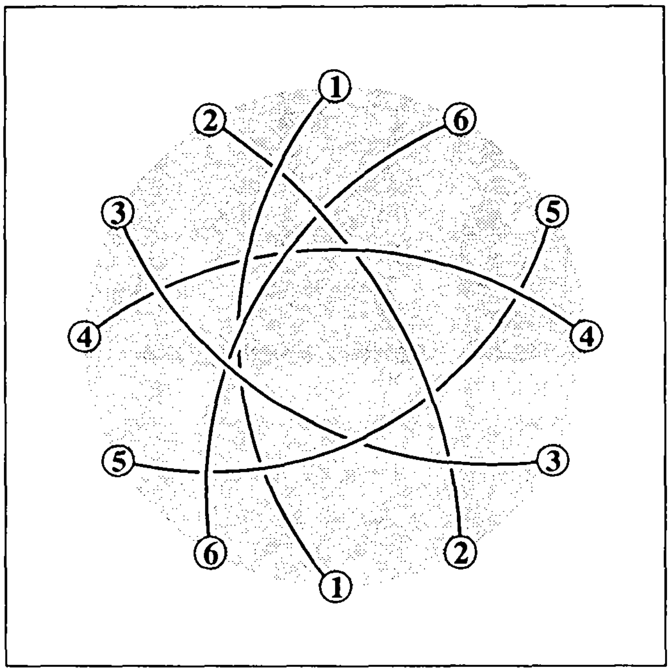


Figure 9. A diagram of six pseudolines.

for every ordered pair A', B' of lines in the arrangement. By Corollary 24, we can regard $\text{under}\,(A', B')$ as the (constant) limit of $\text{toward}\,(\text{after}\,_z(A_t, B_t))$. The structure

$$P \;=\; (L', \text{under}\,)$$

is called the *planar layout* of the configuration L with respect to the projection $\pi_{z,Q}$. It can be regarded as a projection with "depth information", or as the limit of the family $[L_t, \text{toward}\,(\text{after}\,_z()_t)]$. Since $\text{under}\,(A', B')$ is antisymmetric with respect to order of the variables A', B' and independent of their orientation, this definition of under furnishes us with an example of a line diagram. We define a line diagram in the affine plane to be *realizable* or *liftable* if it is the planar layout of a configuration in 3-space. The property of being liftable turns out to be independent of the chosen projection as we now show. Bear in mind that, by definition, no line of a line diagram $\mathcal{H}$ lies at infinity.

Theorem 26. *Suppose we are given a line diagram $\mathcal{D} = (\mathcal{H}, \text{under}\,)$ in the plane Q. If $\mathcal{D}$ is liftable with respect to some projection $\pi_{z,Q}$ then it is realizable with respect to any other projection $\pi_{z',Q}$. Furthermore, if both projections have the same sign, every realization with respect to z is isotopic to some realization with respect to z'.*

Proof: Let τ be the nonsingular transformation of $\mathcal{P}^3$ that leaves Q pointwise fixed and maps z to z'. Assume that $\mathcal{D}$ is the planar layout of a configuration L with respect to $\pi_{z,Q}$. Then we define

$$L' \;=\; \{\tau(A)\,;\; A \in L\}.$$

Since nonsingular projective transformations preserve incidences, L' is again a configuration in $\mathcal{C}$ with

$$\pi_{z',Q}(L') \;=\; \mathcal{H}.$$

Further, let A, B be a pair of lines of L, and let p be the positive point where the projected lines $\pi(A), \pi(B)$ of $\mathcal{H}$ meet. As usual, we define

$$k_A \;=\; (z \vee A) \wedge B$$
$$k_B \;=\; (z \vee B) \wedge A.$$

We use the notation A', B', k'_A, k'_B for the images in $\mathcal{P}^3$under τ of A, B, k_A, k_B. Then

$$\begin{aligned}\det(\tau) > 0 &\iff \mathbf{sign}\,(Q \wedge (k_A \vee k_B)) = \mathbf{sign}\,(Q \wedge (k'_A \vee k'_B)) \\ &\iff \mathbf{sign}\,(Q \wedge (z \vee p)) = \mathbf{sign}\,(Q \wedge (z' \vee p)).\end{aligned}$$

This implies that

$$\omega(z \vee p, k_A \vee k_B) \;=\; \omega(z' \vee p, k'_A \vee k'_B),$$

and hence that

$$\mathbf{toward}\,(\mathbf{after}\,_z(A, B)) \;=\; \mathbf{toward}\,(\mathbf{after}\,_{z'}(A', B')).$$

We conclude that $\mathcal{D}$ is the planar layout of L' with respect to $\pi_{z',Q}$. Finally, if

$$\mathbf{sign}\,([zQ]) = \mathbf{sign}\,([z'Q])$$

then τ is a positive transformation, and so it is in the same connected component of the Lie group $PGl_3(\mathcal{R})$ as the identity, yielding an isotopy between L and L'. □

Theorem 27. *Let $\mathcal{D}$ be the planar layout of L with respect to $\pi_{z,Q}$, and suppose it is also the planar layout of L' with respect to $\pi_{z',Q}$. If L has at least three lines, the configurations L and L' are isotopic (with labels of lines preserved) if and only if $\pi_{z,Q}$ and $\pi_{z',Q}$ have the same sign.*

Proof: If $[zQ][z'Q] < 0$ then L and L' are certainly non-isotopic, for any triple of lines has opposite chiralities in the two configurations. So suppose that the projections $\pi_{z,Q}$ and $\pi_{z',Q}$ have the same sign. By Theorem 26 we do not lose any generality if we regard the liftings L and L' both with respect to the same projection, which may be taken to be the vertical projection of the euclidean space $\mathcal{R}^3$ upon the xy-plane. We can moreover endow the z-axis with the orientation such that the oriented center of projection gets coordinates $(0,0,1,0)$ (rather than $(0,0,-1,0)$). If

$$\mathcal{H} \;=\; \{H_1, \ldots, H_n\}$$

is the arrangement of the planar layout then we can choose the axes in the xy-plane such that each H_i can be described as

$$H_i \;:\; y \;=\; a_i x + b_i.$$

If L_i is a line in $\mathcal{R}^3$ that vertically lifts H_i then it can be represented by the equations

$$\begin{cases} y &= a_i x + b_i \\ z &= c_i x + d_i \end{cases}$$

and hence by the coordinates (c_i, d_i). So the submanifold $\mathcal{R}$ of $(G_{2,4})^n$ consisting of all line sets in $\mathcal{R}^3$ which vertically project upon $\mathcal{H}$ can be identified with $\mathcal{R}^{2n}$, using one global chart. Next, if $L_i = (c_i, d_i)$ and $L_j = (c_j, d_j)$ are two lines whose vertical projections are H_i and H_j, respectively, then L_i and L_j intersect if and only if

$$(b_j - b_i)(c_i - c_j) + (a_i - a_j)(d_i - d_j) \;=\; 0. \tag{10}$$

Observe that L_i and L_j can never be parallel, because their projections H_i and H_j intersect. Taking the a_i, b_i to be constant (for a fixed planar layout), the $\binom{n}{2}$ homogeneous linear equations (10) define a central arrangement of hyperplanes in $\mathcal{R}^{2n}$. A point of $\mathcal{R}$ is a configuration in $\mathcal{C}$ if and only if it lies in the complement of the union of these $\binom{n}{2}$ hyperplanes. Consequently, the (unbounded) open cells of this central arrangement are exactly the connected components of $\mathcal{R} \cap \mathcal{C}$. On the other hand, two configurations belong to the same cell if and only if they have the same sign pattern for the linear forms in $(*)$, if and only if they determine the same function **under** on $\mathcal{H}$ and hence have the same planar layout. □

Observe that we have even shown the existence of a *linear* isotopy between two liftings of $\mathcal{D}$. Throughout this entire isotopy, the planar layout remains fixed. Indeed, the cells of a hyperplane arrangement are *convex*, so two liftings of the same planar layout can be joined by a *segment* completely contained in some component of $\mathcal{R} \cap \mathcal{C}$.

We close this section by proving that an arbitrary planar layout can be realized as the limit of relatively simple "nearby" configurations. Thanks to Theorems 26 and 27, we are free to assume that all planar layouts arise by vertical (parallel) projection into the xy-plane. An ε-*cylindric model* is a configuration L of lines and a cylinder

$$C = D \times [-\varepsilon, \varepsilon]$$

enclosing a disc of finite radius in the xy-plane, such that

(1) the lines of L pass through the interior of C, but do not meet either the top or bottom of the cylinder.

(2) the vertical directrix K_{AB}, for any pair A, B of lines in L, meets the disc D.

The relation between an ε-cylindric model and its planar layout relative to vertical projection is particularly simple:

$$\textsf{under}\,(A', B') = +1 \text{ if and only if the line } A \text{ passes under the line } B \text{ in the cylinder } C\ .$$

Theorem 28. *For any $\varepsilon > 0$ and any planar layout $\mathcal{D}$, there is an ε-cylindric model L such that $\mathcal{D}$ is the limit of a vertical projection of L.*

Proof: Given $\varepsilon > 0$ and any planar layout $\mathcal{D}$, realize $\mathcal{D}$ as the limit of the vertical projection of a configuration L. Let D be a circular disc containing all intersection points of pairs of lines in $\mathcal{D}$. Let h be the maximum absolute value of the height of any line in L over the (compact!) boundary of the disc D. Transform the entire space by scaling the vertical dimension by a factor of ε/h, thereby creating a configuration L' such that every line passes through the interior of the cylinder $D \times [-\varepsilon, \varepsilon]$, but no line meets the top or bottom of the cylinder. The limit of the vertical projection of L' is equal to $\mathcal{D}$. □

Although it is not always easy to decide whether a line diagram is a planar layout of a configuration of lines, there are some situations in which it is trivial. First, it is possible that there is some linear order, say $L_1, L_2, \ldots, L_n$, of the lines of the diagram, such that

$$\textsf{under}\,(L_i, L_j) = +1 \quad \text{for all } i < j.$$

In that case, it suffices to realize line L_i parallel to the xy-plane, at height i. We call such a line diagram *stacked*. For non-stacked diagrams, there is often an almost equally easy construction. We say a line L is *shellable* from a line diagram H if and only if there exists a finite point p on L, not a crossing point, such that L crosses *over* all lines of H on one side of the point p in the affine plane, *under* all lines of H on the other side of the point p. We say the diagram H is *shellable* if and only if there is a linear order, say $L_1, L_2, \ldots, L_n$, of the lines of the diagram, such that for each $i, 4 \leq i \leq n$, the line L_i is shellable from the diagram

$$H_{i-1} = \{L_1, \ldots, L_{i-1}\}.$$

The definitions apply in the context of pseudoline diagrams. For instance, the pseudoline diagram in Figure 9 is shellable relative to the order $L_5, L_2, L_3, L_6, L_4, L_1$.

Theorem 29. *Any shellable line diagram is realizable.*

Proof: We proceed by induction on the number of lines in the diagram. Since the definition of "shellable" is itself inductive, it suffices to prove that if the line L_n is shellable in a diagram $H = \{L_1, \ldots, L_n\}$, and if the diagram $H_{n-1} = \{L_1, \ldots, L_{n-1}\}$ is realizable, then so is H. Let p be a point on L_n which separates its under-crossings from its over-crossings. By Theorem 28, we may realize the diagram H_{n-1} by an ϵ-cylindrical model, and choose a vertical cone with small radius, with center p, not meeting any line of the cylindrical model of L_{n-1}. Within the surface of this cone there is a unique line which projects to L_n and has the correct values of $\mathbf{under}\,(L_i, L_n)$. So the induction is established, and any shellable diagram is realizable. □

8. Link chirotopes

Let L be a configuration of n lines in $\mathcal{P}^3$, and z any point not incident with any of those lines. Associated with the perspective view of L from z as center (Section 4), we find a chirotope [Bj], which we may enrich with linking information to form a combinatorial structure which we call a "link chirotope". The underlying chirotope can be thought of either as the system Q_z of oriented planes $Q_L = z \vee L$, or as the family T_z of oriented great circles T_L obtained by intersecting the oriented planes Q_L with the unit sphere S having point z as center (see Section 11).

In Section 4, we defined the linking number only for pairs of *skew* lines. In what follows, it will be convenient to include the value 0 for the linking number of *incident* lines. This becomes consistent with our definition

$$\mathbf{link}\,(A, B) = \mathbf{sign}\,(A \vee B)$$

once we enlarge the set of "signs" to the ternary integers $\{-1, 0, 1\}$.

For any three lines A, B, C in a configuration L, and any point z in $\mathcal{P}^3$, which we shall call the *center*, not incident with any of those lines, we define $\mu_z(A, B, C)$ to be the sign of the scalar w in the expression

$$(z \vee A) \wedge (z \vee B) \wedge (z \vee C) = wz.$$

There is one degeneracy possible. When the point z lies on the ruled quadric surface (a hyperboloid of one sheet, or a hyperbolic paraboloid) generated by three of the lines A, B, C, then the three planes $z \vee A, z \vee B, z \vee C$ will have a *line* in common (the directrix through z, meeting all lines of the ruling). Under these conditions, the meet of the three planes will be zero, and the directrix through z meeting any two of the three lines will also meet the third. In this case, we accept that $\mu_z(A, B, C) = 0$.

The combinatorial structure $(|L|, \mu_z, \mathbf{link})$, consisting of the abstract set $|L|$, the assignment,

$$\{A, B\} \mapsto \mathbf{link}\,(A, B)$$

to each unordered pair A, B of $|L|$, together with the assignment

$$(A, B, C) \mapsto \mu_z(A, B, C)$$

to each ordered triple A, B, C of $|L|$, we call the *link chirotope* $Q_z(L)$.

Abstractly, a rank 3 *link chirotope* $(A, \mu, \mathbf{link})$, is a chirotope (A, μ) on a finite set A, together with an *arbitrary* sign function $\mathbf{link}$ on pairs of elements of A. That is, as defined in [Bj], μ is a sign function on $\binom{A}{3}$ satisfying, for every six elements a, b, c, d, e, f in A:

$$\begin{array}{lrcl} \text{if} & \mu(abc)\,\mu(def) \;=\; \mu(acd)\,\mu(bef) & = & +1 \\ \text{and} & \mu(abd)\,\mu(cef) & = & -1 \\ \text{then} & \mu(bcd)\,\mu(aef) & = & +1 \end{array}$$

This chirotope axiom is simply the "combinatorial content" of the first-order syzygy

$$[abc][def] - [abd][cef] + [acd][bef] - [bcd][aef] = 0.$$

There are rank 3 chirotopes that are not realizable as chirotopes of arrangements of lines in the plane. Instead, they arise from configurations of pseudolines [Gr]. So there are link chirotopes whose underlying chirotope is not realizable. Furthermore, there are link chirotopes whose underlying chirotope is realizable, but which are not realizable due to conflicts in the 'weaving pattern" specified by the sign function **link**. In what follows, we consider only those link chirotopes which arise from perspective views of configurations of lines.

Theorem 30. *Let* $Q_z(L) = (|L|, \mu_z, \textbf{link})$ *be a link chirotope, and let* Q *be a plane not containing the point* z. *Then the sign function* μ_z *is the sign of a bracket on the dual space of lines in the plane* Q.

Proof: Without loss of generality, choose coordinates for Q such that $[zQ] = +1$. For any line $L \in L$, let $L' = \pi_{z,L} = (z \vee L) \wedge Q$. By Proposition 14, $z \vee L' = z \vee L$, so for any lines $A, B, C \in L$,

$$\mu_z(A, B, C) = \mu_z(A', B', C').$$

Since the coefficient of z in the meet

$$(z \vee A') \wedge (z \vee B') \wedge (z \vee C')$$

is a skew-symmetric 3-linear function on the rank 3 space of lines in Q, it is a bracket on the dual space Q^*. The function μ_z is the sign of that bracket. □

Theorem 31. $(|L|, \mu_z)$ *is a chirotope of rank 3, namely, the chirotope determined by any positive projection* $L' = \pi_{z,Q}(L)$ *onto any plane* Q.

Proof: Let $\pi_{z,Q}$ be a positive projection from z as center, onto a plane Q. Without loss of generality, let $[z, Q] = +1$. For any line $L \in L$, let $L' = \pi_{z,Q}(L)$. The sign of a basis A', B', C' in the chirotope of the plane configuration L' is the sign of the bracket of those lines in the dual vector space Q^*. By Theorem 30, this sign function is equal to μ_z. □

For instance, if $z = (0, 0, 1, 0)$ and $Q = (1, 0, 0, 0)$, so $\pi_{z,Q}$ is vertical projection onto the xy-plane, then the embedding is given by

$$\begin{array}{rcc} a = (a_1, a_2, a_3) & \longrightarrow & (a_1, a_2, 0, a_3) \\ A = (A_{12}, A_{13}, A_{23}) & \longrightarrow & \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ A_{12} & 0 & A_{13} & 0 & A_{23} & 0 \end{pmatrix} \end{array}$$

and

$$\begin{array}{rcl} (z \vee A) \wedge (z \vee B) \wedge (z \vee C) & = & (A_{12}B_{13}C_{23} - \ldots - A_{23}B_{13}C_{12})\ z \\ & = & (A \underset{2}{\wedge} B \underset{2}{\wedge} C)\ z \end{array}$$

To any ray K through z, we can then associate its *sign vector* " **side** ", namely, the assignment

$$\textbf{side}_K(L) = \textbf{link}(K, L)$$

for all $L \in |L|$.

Proposition 32. $\textbf{side}_K$ *is the sign vector customarily associated with the point* K *on a facet of the chirotope* (L, μ_z).

Proof: Let Q be a plane not containing the point z, and such that $[zQ] = +1$. The sign vector of a point p in the chirotope of a plane configuration of oriented lines in Q is the sign pattern of the scalars $[pL]_2$, where $[\ ]_2$ is the bracket (rank 3) in the plane Q. For any oriented line K through z, let $p_K = K \wedge Q$. By Theorem 1 of Section 2, $zp = [zQ]K = K$. The function $\textbf{side}_K = \textbf{link}(K, L)$ is the sign of the bracket $K \vee L = [zpL]$. For fixed z, the bracket $[zpL]$ is symmetric, bilinear on the product space $Q \times Q^*$, and arises from a bracket (3-linear, skew-symmetric) on the space Q. Up to an overall scalar multiple, this bracket is equal to $[pL]_2$. (The sign of the bracket determines the "half-space" of the oriented hyperplane $\pi_{z,Q}(L)$ in which lies the point p.) If the ray K does not meet any of the lines in L, then the sign vector has no entries equal to 0, so it determines a *facet* of the chirotope. □

Note: If K is a ray toward a positive point p, then $\textbf{side}_K(L)$ is $+1$ for those lines L such that the point p is on the side of the oriented plane $z \vee L$ indicated by the "left-hand rule". That is, in the half-space containing the point p, the plane $z \vee L$ is oriented CW around the point z.

For three oriented skew lines A, B, C not meeting z, the planes

$$Q_A = z \vee A, \quad Q_B = z \vee B, \quad Q_C = z \vee C,$$

separate the affine space of finite points into eight triangular cones, or *octants*[8], centered at z. Each cone is bounded by portions of the three planes, meeting along the three directrices K_{AB}, K_{AC}, K_{BC}, where, for instance,

$$K_{AB} \;=\; (z \vee A) \wedge (z \vee B) \;=\; \textbf{turn}_z(A, B)$$

The perspective view from z into any one of these octants reveals one of two situations. The lines orient the faces of the triangular cone either cyclically (head-to-tail, in a cycle) or acyclically. In the latter case, we may distinguish the directrices as being either *source* (at two tails), *sink* (at two heads), or *through* (at a head and a tail).

Proposition 33. *For any three oriented skew lines A, B, C, not meeting a point z in $\mathcal{P}^3$, the perspective view into the eight octants formed by the planes $z \vee A$, $z \vee B$, $z \vee C$ will invariable reveal two cyclically oriented octants (one CCW, one CW) and six acyclically oriented octants.*

Proof: This comes down to noticing that if a directrix is "source" or "sink" in one octant, it will be "through" in an octant reached from the first by crossing one plane. □

For any oriented skew lines A, B not meeting the point z, let

$$d_{AB} \;=\; \textbf{turn}_z(A, B) \wedge (-1, 0, 0, 0)$$

be the direction, the oriented point where their **turn** meets the plane at infinity.

Theorem 34. *Let A, B, C be skew lines not meeting a point z in $\mathcal{P}^3$. The sign $\mu(ABC)$ is positive if and only the circular order (ABC) agrees with the orientation of any cyclic quadrant, or with the orientation (source, sink, through) of any acyclic quadrant.*

Proof: First, we consider the case where the lines A, B, C, are oriented CCW, in the circular order (ABC), around the octant in question. Then we can choose positive points a, b, c, d, e, f, along the three rays bounding the quadrant, in such a way that

$$\begin{gathered} zab \;=\; zcd \;=\; zef \;=\; 0, \\ z \vee A \;=\; zbc, \quad z \vee B \;=\; zde, \quad z \vee C \;=\; zfa, \\ \textbf{sign}\,(zbce) \;=\; \textbf{sign}\,(zadf) \;=\; +1, \end{gathered}$$

[8] Projectively: four *footballs* each with three digonal faces, and a singular point, z, as sole vertex.

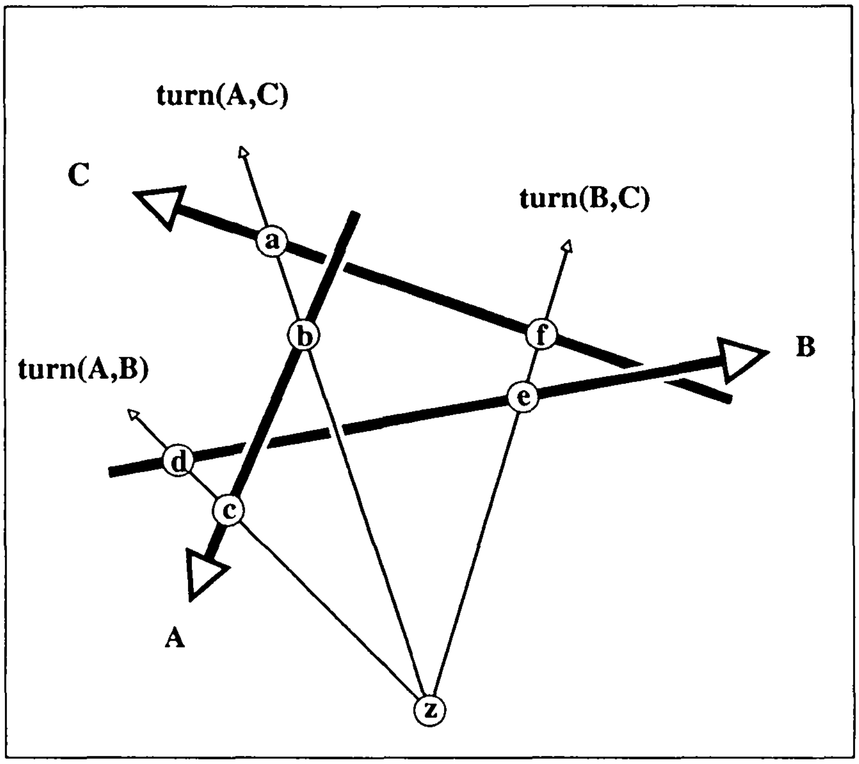


Figure 10. One octant: the order of the points on each **turn** does not effect the value of μ.

as in Figure 10. The order of the points a, b on **turn** $_z(C, A)$, of c, d on **turn** $_z(A, B)$, and of e, f on **turn** $_z(B, C)$, are immaterial to the proof. Then

$$(z \vee A) \wedge (z \vee B) \wedge (z \vee C) \;=\; zbc \wedge zde \wedge zfa$$
$$=\; zd[zbce] \wedge zfa \;=\; [zbce][zadf]z \;\sim\; z$$

Any re-orientation of the lines A, B, C, and thus of the planes

$$z \vee A, \quad z \vee B, \quad z \vee C,$$

is accomplished by exchanging either

$$b \leftrightarrow c \text{ or } d \leftrightarrow e \text{ or } f \leftrightarrow a$$

in the above paragraph. Each such change reverses the sign of the expression $zbc \wedge zde \wedge zfa$.

Say the circular order (ABC) agrees with the orientation of a cyclic quadrant. If the orientation of one line is reversed, then the quadrant becomes acyclic, with the circular order (ABC) opposite to that of (source, sink, through). If, on the other hand, the circular order (ABC) agrees with the order (source, sink, through) of an acyclic quadrant, the reversal of orientation of any one line creates either a cyclic quadrant with orientation opposite to the circular order (ABC), or another acyclic quadrant with orientation opposite to the circular order (ABC). □

By Theorem 9 of Section 4, **after** and **turn** determine **link**. If we arbitrarily select the orientations of two lines, the remaining orientations are given by the sign function μ of the chirotope. The orientations determined, we have the sign functions **after** and **turn**. The equivalent information furnished by **after** is particularly helpful in the situation where a ray K through the center of perspective z meets several lines of the configuration. The points at which the lines of the configuration meet the ray are *linearly ordered* along the ray. The sign function $\textbf{after}_z$ records that linear order.

9. Moves on line diagrams

In this section we study types of isotopy weaker than rigid isotopy of lines, stronger than flexible isotopy of links. We believe that this investigation will lead us to the construction of examples of non-isotopic line configurations that cannot be topologically distinguished. The existence or non-existence of such configurations remains one of the main open problems concerning configurations of projective lines.

Let L_0 and L_1 be two isotopic line configurations, and let $\{L_t; 0 \leq t \leq 1\}$ be an isotopy that connects them. The i^{th} line $(L_t)_i$ of the configuration L_t, corresponding to the line L_i at time t, we denote by $L_i(t)$. Choose a plane at infinity that is transverse to both given configurations, so that L_0 and L_1 are seen in a common affine subspace $\mathcal{R}^3$. Then choose a projection $\pi_{z,Q}$ proper with respect to both configurations, that is: no two lines of either configuration are parallel in projection. Let D_0 and D_1 be the resulting planar layouts. What diagram changes can take place under an isotopy of lines?

In order to represent a planar layout by a line diagram in the affine plane Q, we can moreover assume that a projection $\pi_{z,Q}$ has been selected so that both configurations are *regular.* The projections $\pi_{z,Q}(L_1)$ and $\pi_{z,Q}(L_2)$ are then regular line arrangements in the affine plane: no line degenerates to a point, no two lines are parallel, no three lines meet at a point. Of course, we cannot require regularity for the whole continuous family of configurations L_t. It is usually necessary to have configurations with singular projections during any isotopy that changes the combinatorial type of the underlying diagrams.

In what follows, I denotes the line at infinity in the plane Q. From the definition of regularity it follows that a configuration L is singular if and only if it belongs to one of the following three types.

- $||$ There exists a pair $\{L_i, L_j\}$ of L such that $\mu_z(L_i, L_j, I) = 0$. This pair becomes parallel in the projection, which is not proper anymore with respect to the configuration.
- $\not\!\!\times$ There exists a line through z that intersects at least three lines of the configuration. These latter lines become concurrent in the projection. In this case, z lies on the ruled surface spanned by three lines L_i, L_j and L_k of L, so $\mu_z(L_i, L_j, L_k) = 0$.
- $\bullet$ At least one of the lines contains z. In this case the projection is not the union of n lines, for at least one line projects onto a point.

It is our purpose to show that such singularities can be removed or isolated, as the case may be. The situation is summed up as follows:

Theorem 35. *If L_0 and L_1 are two isotopic line configurations then they can be connected by an isotopy $\{L_t; 0 \leq t \leq 1\}$ such that*

(1) for each $t \in [0, 1]$ no line of L_t lies at infinity;

(2) at no time t there is an $\bullet$-singularity;

(3) there are only a finite number of moments t for which L_t is a configuration of type $||$ or type $\not\!\!\times$, and during such a time t exactly one singularity occurs.

In order to prove Theorem 35, we appeal to the following result from differential geometry.

Theorem 36. *Let M be a smooth manifold, let A be a submanifold of codimension 1, and let B be a submanifold of codimension 2. If p and q are two points in the same connected component of M, and* *neither p nor q lie in the submanifold B, then there exists a continuous path $\pi : [0,1] \to M$ connecting p to q such that*

(1) π avoids B: $(\forall t \in [0,1]), \pi(t) \notin B$,

(2) π intersects A transversally a finite number of times: $(\#(A \cap \{\pi(t); 0 \le t \le 1\} < \infty)$.

Proof: In the particular case that $M = \mathcal{R}^m$ and that A and B are linear subspaces then we can use the concatenation of two line segments not lying in the hyperplane A and not meeting the coline B to join two arbitrary points p and q in M.

In general, we can choose local coordinates for M such that A and B appear locally as linear subspaces of $\mathcal{R}^m$. So for each point $x \in M$, there exists a neighborhood V_x of x, a chart $\Psi_x : V_x \to \mathcal{R}^m$, and linear subspaces A_x and B_x of $\mathcal{R}^m$ such that for each $y \in V_x$,

$$y \in A \text{ (resp, } B) \text{ if and only if } \Psi_x(y) \in A_x \text{ (resp, } B_x).$$

Suppose now we start with an arbitrary continuous path $h : [0,1] \to M$ with $h(0) = p$ and $h(1) = q$. Because $h[0,1]$ is compact in M we can cover it with a finite number of these particular chart domains V_x:

$$h([0,1]) \subset V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_k}$$

where the labels are chosen such that $V_{x_i} \cap V_{x_{i+1}} \neq \emptyset$. We select an arbitrary point y_i in each set $V_{x_i} \cap V_{x_{i+1}}$ such that $y_i \notin A \cup B$. Furthermore, we take $y_0 = h(0), y_k = h(1)$. We construct in each $\phi_{x_i}(V_{x_i}) \subset \mathcal{R}^m$ a continuous path π_i that connects y_{i-1} to y_i, avoiding the subspaces B_{x_i} and intersecting the subspace A_{x_i} only a finite number of times. Finally, in the manifold M we glue all the preimages, setting

$$\pi \;=\; \Psi_{x_1}^{-1}(\pi_1) \cup \Psi_{x_2}^{-1}(\pi_2) \cup \;\ldots\; \Psi_{x_k}^{-1}(\pi_k)$$

□

Proof: *(of Theorem 35)*

The submanifold A of the configuration space C consisting of those configurations having a $\|$ - or a $\not\vert$ - singularity has codimension 1. The submanifold B consisting of those configurations in C having either a •-singularity, or one line at infinity, or at least two singularities, has codimension 2. So the assertion of Theorem 35 is a consequence of Theorem 36. □

From now on, each isotopy of lines will be assumed to satisfy the conditions of Theorem 35. We can thus restrict all configurations during an isotopy to lie within some fixed affine space $\mathcal{R}^3$, their planar layouts being defined relative to a fixed projection. On the other hand, let $\mathcal{D}_0$ and $\mathcal{D}_1$ be regular planar layouts with underlying regular arrangements H_0 and H_1, respectively. Let D denote the submanifold of $(G_{2,4})^n$ consisting of all planar configurations lying in some fixed plane Q. Let $\{H_t; 0 \le t \le 1\}$ be a continuous path in D connecting H_0 and H_1. We call $\{H_t\}$ a *homotopy between regular arrangements* if

(1) There are only a finite number of times $t \in [0,1]$ such that H_t is non-regular;

(2) If H_t is non-regular then either exactly one triple of lines is concurrent or exactly one pair of lines is parallel. The former is called a $\not\vert$-singularity and the latter a $\|$-singularity of the homotopy.

Any pair of regular arrangements can be connected by such a homotopy: we select a parametrized family of plane arrangements, and obtain the isolation of singularities, (1) and (2), by techniques similar to those in the proof of Theorem 36. If H_0 is a planar arrangement of oriented lines, then this orientation can be chosen continuously along a continuous path, every arrangement H_t of the homotopy inheriting an orientation from H_0. In this way we orient the whole homotopy.

Suppose now we have such an oriented homotopy, and assume H_t is not a $\|$-singularity. We can define $\textbf{turn}_t(i,j) \in \{-1,+1\}$ for each pair $\{i,j\} \in \{1,\ldots,n\}$ by the "CW-CCW rule" for oriented lines i and j in H_t (Corollary 11). That is, we set $\textbf{turn}_t(i,j) = +1$ if and only if line i of H_t requires a CCW turn of less than 180° to match the orientation of line j of H_t. Even if H_t is a $\|$-singularity, we can define $\textbf{turn}_t(i,j)$, as long as the singularity is not due to parallelism of the lines $\{i,j\}$. For fixed $\{i,j\}$, the sign $\textbf{turn}_t(i,j)$ is a function of t. The subset of $[0,1]$ for which $\textbf{turn}_t(i,j)$ is defined is the *domain* of that function.

Two configurations $\mathcal{D}_0$ and $\mathcal{D}_1$ are called *homotopic as line diagrams* if there exists a homotopy $\{H_t; 0 \le t \le 1\}$ between the underlying regular arrangements H_0 and H_1, and a family $\{\textbf{under}_t;\ \text{for}\ t \in [0,1]\}$ of antisymmetric sign functions

$$\textbf{under}_t(i,j) = -\textbf{under}_t(j,i) \in \{-1,+1\}$$

defined for each t for which H_t is not a $\|$-singularity due to $\{i,j\}$, such that

(1) $\mathcal{D}_0 = (H_0, \textbf{under}_0)$ and $\mathcal{D}_1 = (H_1, \textbf{under}_1)$.

(2) For each orientation of the homotopy $\{H_t\}$ and for each pair $\{i,j\}$, the sign

$$\textbf{link}_t(i,j) = \textbf{turn}_t(i,j) \times \textbf{under}_t(i,j)$$

is constant for each t that is not a $\|$-singularity for the pair $\{i,j\}$.

(3) If t is a $\not\times$-singularity then the relation $\textbf{under}_t$ is a partial order (transitive) on the three lines involved in the singularity.

Lemma 37. *If $\{\mathcal{D}_t; 0 \le t \le 1\}$ is a homotopy of line diagrams then for each pair $\{i,j\}$ the function $\textbf{under}_t(i,j)$ is constant on the components of its domain of definition, that is, it can only change at $\|$-singularities.*

Proof: If $z \in \mathcal{R}^3$ is not in the plane Q of the homotopy, where $\pi_{z,Q}$ is a positive projection, and if $L_i'(t)$ denotes the position of the i^{th} line in H_t, then by the definition of $\textbf{turn}_t$ and by Corollary 11

$$\textbf{turn}_t(i,j) \quad = \quad \textbf{toward}\,(\textbf{turn}_z(L_i'(t), L_j'(t))).$$

It follows from Theorem 18 that for each pair $\{i,j\}$, the sign $\textbf{turn}_t(i,j)$ is constant on the components of its domain. The assertion is a consequence of the invariance of $\textbf{turn}_t(i,j) \times \textbf{under}_t(i,j)$, required in the definition of diagram homotopy. □

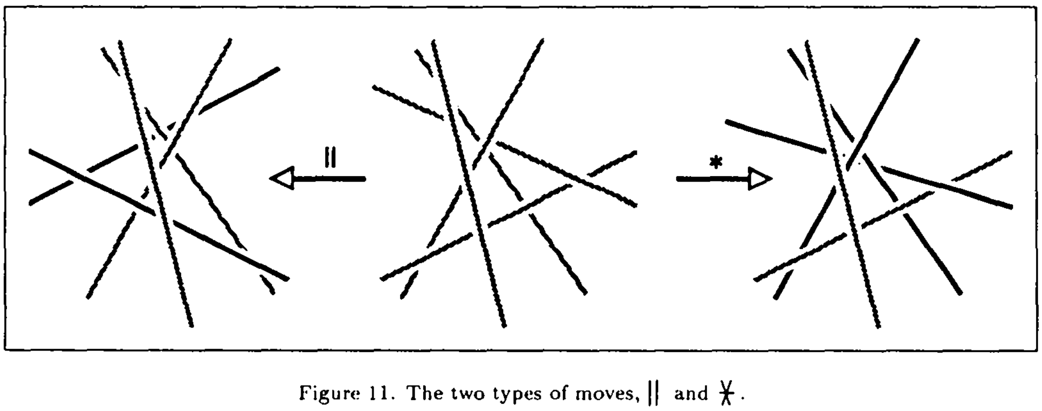


Figure 11. The two types of moves, $\|$ and $\not\times$.

If t is a ¥-singularity of lines $\{i, j, k\}$ then in a connected neighborhood of t that avoids $\|$-singularities the lines with these labels make a stack (rather than a vortex). This means that during a homotopy of line diagrams the only combinatorial changes can occur at the singularities, according to two types of diagram changes, called *diagram moves.* These types are called $\|$ and ¥ after the type of singularity at which they occur, and are given in Figure 11. (Compare the ¥-move with the Reidemeister-3 move in knot theory.)

Diagram moves can be described in a purely combinatorial fashion, as actions on combinatorial line diagrams. As an action on the underlying affine regular arrangement, $\|$ "flips" an intersection to the "other side", while ¥ causes a "triangle-move". Notice that $\|$ does not modify the combinatorial type as projective arrangement; it merely changes the choice of the line at infinity. The "weaving" structure **under** is only affected by $\|$. If the combinatorial type of two line diagrams can be transformed to each by a sequence of moves of type $\|$ or ¥ then we say that the diagrams are *equivalent.* If we insist that the intermediate pseudoline diagrams are all stretchable then we call the line diagrams *rigidly equivalent.* It is clear that two homotopic line diagrams are always rigidly equivalent.

Finally, we say that two lines diagrams are *lift-homotopic* if they can be transformed to each other by a diagram homotopy

$$\{\mathcal{D}_t;\ \ 0 \le t \le 1\} \ \ = \ \ \{(H_t, \mathbf{under}\ _t);\ \ 0 \le t \le 1\}$$

such that at any time t the diagram is realizable, that is, $(H_t, \mathbf{under}\ _t)$ is a planar layout for all $t \in [0,1]$. This completes the terminology necessary to state the equivalent of Reidemeister's Theorem for line isotopy. We first establish a technical result.

Lemma 38. *Let D denote the submanifold of $(G_{2,4})^n$ consisting of all configurations lying in some fixed plane Q. If H is a planar configuration in D, and if L is a configuration in C that projects upon H then there exists a neighborhood W of H in D such that each $H' \in W$ is the projection of a configuration L' that is isotopic to L.*

Proof: Let z denote the center of projection. We select a plane P_i through each member L_i of L such that $z \notin P_i$. As usual, $L_i' \in H$ denotes the projected image of L_i. Then we have

$$L_i = (z \vee L_i') \wedge P_i, \tag{11}$$

for $i = 1, \ldots, n$. Next, for each pair $\{i, j\}$ we define the directrix

$$K_{ij} = (z \vee L_i') \wedge (z \vee L_j'),$$

whose Plücker coordinates are continuous functions in those of L_i' and L_j'. Further, we define

$$A_{ij} = P_i \wedge P_j.$$

These "meets" K_{ij} and A_{ij} are well-defined projective lines, since L lies in the configuration space C. If we introduce $6n$ variables for the homogenous coordinates of $L_1', \ldots, L_n'$, we can consider the subvariety $V(F)$ of $(G_{2,4})^n$ defined by the following homogeneous polynomial identity

$$F = \Pi_{i,j}(A_{ij} \vee K_{ij}).$$

If we substitute the coordinates of the given H then we get $F \neq 0$ since

$$A_{ij} \vee K_{ij} \ \ = \ \ 0 \iff L_i \vee L_j \ \ = \ \ 0.$$

Conversely, if $H' = \{L_1', \ldots, L_n'\}$ is a point in $(G_{2,4})^n \setminus V(F)$ then it induces a configuration $L = \{L_1, \ldots, L_n\}$ in C by Equation (11). If N is a connected neighborhood of H in the open $(G_{2,4})^n \setminus V(F)$

then $W = N \cap D$ is the desired neighborhood in D. Indeed, if $H' \in W$ and if $\{H(t); 0 \leq t \leq 1\}$ is a continuous path in W connecting $H = H(0)$ with $H' = H(1)$, then we can lift it to an isotopy $\{L_t; 0 \leq t \leq 1\}$ by

$$L_i(t) = (z \vee L_i'(t)) \wedge P_i.$$

□

Theorem 39. *Let L_0 and L_1 be two configurations in $\mathcal{C}$. Choose the plane at infinity transverse to both configurations, and assume that they are regular with respect to the projection $\pi_{z,Q}$, leading to the line diagrams $\mathcal{D}_0 = \pi_{z,Q}(L_0)$ and $\mathcal{D}_1 = \pi_{z,Q}(L_1)$. Then L_0 is isotopic to L_1 if and only if $\mathcal{D}_0$ is lift-homotopic to $\mathcal{D}_1$.*

Proof: First, suppose that L_0 and L_1 are isotopic configurations. If $\{L_t; 0 \leq t \leq 1\}$ is an isotopy with the properties of Theorem 35, then the projections

$$H_t = \pi_{z,Q}(L_t)$$

define a homotopy $\{H_t\}$ between the regular arrangements H_0 and H_1. If L_t is an S_1-singularity with respect to $\{L_i(t), L_j(t)\}$, then t is a $||$-singularity with respect to $\{L_i'(t), L_j'(t)\}$. So, if t is not a $||$-singularity with respect to $\{i, j\}$, we can define

$$\textbf{under}_t(i,j) = \textbf{under}\,(L_i'(t), L_j'(t)).$$

In order to prove that $\{H_t, \textbf{under}_t; 0 \leq t \leq 1\}$ is a lift-homotopy, it suffices to verify properties (2) and (3) in the definition of diagram-homotopy.

To prove property (2), we assume an arbitrary orientation of the homotopy $\{H_t\}$, and consider the consistent (lifted) orientation of the isotopy $\{L_t\}$. By Theorem 16, and by the definition of $\textbf{link}_t(i,j)$ for a diagram-homotopy, we see that

$$\textbf{link}_t(i,j) = \textbf{link}\,(L_i(t), L_j(t)).$$

Since **link** is an invariant of isotopy of oriented lines, $\textbf{link}_t(i,j)$ is an invariant.

Property (3) holds because at any $\not\times$-singularity due to a triple of lines $\{L_i(t), L_j(t), L_k(t)\}$ the common directrix through z of this triple meets them in some linear order, which forces the relation **under** to be transitive on that triple.

Conversely, suppose that the planar layouts of the two given configurations are lift-homotopic. Say that the lift-homotopy is given by $\mathcal{D}_t = (H_t, \textbf{under}_t)$. Fix a realization L_t for some $\mathcal{D}_t$, where t is not a $||$-singularity. By Lemma 38 we can find an open neighborhood V_t of t in $[0,1]$ and an isotopy $\{L_s; s \in V_t\}$, such that

$$H_s = \pi_{z,Q}(L_s)$$

for each $s \in V_t$. Furthermore, by definition of diagram-homotopy on one hand, and the invariance of **link** under isotopy of oriented lines on the other hand, we get that

$$\textbf{link}\,(L_i(s), L_j(s)) = \textbf{link}\,(L_i(t), L_j(t)) = \textbf{link}_t(i,j) = \textbf{link}_s(i,j)$$

for each $s \in V_t$ that is not a $||$-singularity with respect to $\{i, j\}$. If we combine this with

$$\textbf{turn}_s(i,j) = \textbf{toward}\,(\textbf{turn}_z(L_i(s), L_j(s)))$$

then we obtain that

$$\textbf{under}_s(i,j) = \textbf{under}\,(L_i'(s), L_j'(s)).$$

To sum up, if $s \in V_t$ is not a $\|$-singularity then $\mathcal{D}_s$ is the planar layout of L_s. By compactness of $[0,1]$ we can select a finite number of times $0 \leq t_1 < \ldots < t_k \leq 1$, none of which is a $\|$-singularity, and open neighborhoods V_{t_i} of t_i, such that

(1) $[0,1] = V_{t_1} \cup \ldots \cup V_{t_k}$;

(2) for each $i = 1, \ldots, k$ we have an isotopy $\{L^{(i)}(s); s \in V_{t_i}\}$, where $\mathcal{D}_s^{(i)}$ is the planar layout of $L_s^{(i)}$ if $s \in V_{t_i}$ is not a $\|$-singularity.

So if $s \in V_{t_i} \cap V_{t_{i+1}}$ is not a $\|$-singularity then $L_s^{(i)}$ and $L_s^{(i+1)}$ are liftings of the same planar layout $\mathcal{D}_s$, and hence, by virtue of Theorem 27, are isotopic. The assertion finally follows by glueing all these isotopies together. □

10. Link Invariants

There is an intrinsic connection between families of n mutually skew lines in projective 3-space, and families of n mutually linked circles. In this section we compute the explicit equations of these circles as a function of the line coordinates of the given family of lines. Furthermore, we give geometric properties of this transformation, which make it easy to visualize.

Much progress has been made these past few years in the study of isotopy invariants of links and knots ([Ka_{1-4}], [Fr], [Jo], [Li]). In particular, there is an extremely attractive theory of polynomial invariants of such objects. Recently, this theory has been further extended to embrace Tutte-Grothendieck invariants of graphs, and connections with statistical physics have emerged ([Ka_4]).

In order to make contact with the theory of links, we carry out a two-stage procedure. First we use a central projection in 4-space to replace every point p in projective 3-space by a *pair* p', p'' of antipodal points on the unit 3-sphere S^3 in real 4-space, $\mathcal{R}^4$. The sphere S^3 is a semi-projective space, the spherical model of "oriented projective 3-space", as developed in [St].

Then, using as center one of the semi-projective points o' representing the origin o of $\mathcal{P}^3$, we stereographically project $S^3 - \{o'\}$ into $\mathcal{R}^3$. This mapping can be extended to all of S^3 by mapping o' to a single point "∞", which completes $\mathcal{R}^3$ to a Riemann 3-sphere. Under the composite of these two projections

$$\mathcal{P}^3 \overset{\rightarrow}{\underset{\rightarrow}{}} (S^3 \in \mathcal{R}^4) \rightarrow \mathcal{R}^3 \cup \infty,$$

the "finite" points of the original projective 3-space are doubly represented by points outside and by points inside the unit 2-sphere. In particular, the origin o is represented by the origin $o' = (0,0,0,1)$ of $\mathcal{R}^3$ and by the added point $o'' = \infty$. We shall show that lines in $\mathcal{P}^3$ are mapped to circles lying in planes through the origin and intersecting the unit sphere in antipodal points. Furthermore, pairs of disjoint lines in $\mathcal{P}^3$ are mapped to *linked circles*.

The above construction is easily described using homogeneous coordinates in projective 4-space. With each point p in projective 3-space $\mathcal{P}^3$, we associate the 1-dimensional subspace of homogeneous coordinate representations $(p_1, \ldots, p_4)$ of p, then normalize these representations by dividing by their norm $|p| = (p_1^2 + \ldots + p_4^2)^{1/2}$. Each point p in $\mathcal{P}^3$ is replaced by a pair of distinct semi-projective points[9]

$$p' = (p_1, \ldots, p_4)/|p|$$

$$p'' = (-p_1, \ldots, -p_4)/|p|,$$

on the unit 3-sphere in $\mathcal{R}^4$. The construction is shown in Figure 12. Using homogeneous coordinates to represent points of $S^3 \subseteq \mathcal{R}^4 \subseteq \mathcal{P}^4$, we find

$$p' = (p_1, \ldots, p_4, |p|),$$

[9] Observe that such semiprojective points with distinct coordinates are distinct; no non-trivial equivalence remains.

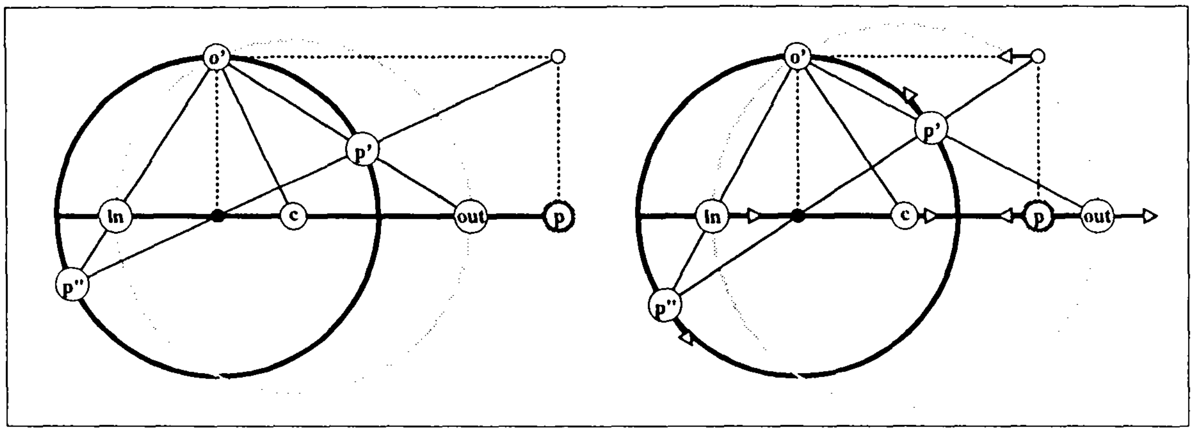


Figure 12. The pair $\gamma_{in}, \gamma_{out}$ of images under the Klein map, for two positions of a point p.

$$p'' = (-p_1, \ldots, -p_4, |p|).$$

The origin "o" of $\mathcal{R}^3$ is mapped to $o' = (0,0,0,1,1)$ and to $o'' = (0,0,0,-1,1)$. We use this point o' as center (or "north pole") of stereographic projection onto the "equatorial" subspace F with equation $x_4 = 0$, and coordinates

$$\begin{pmatrix} 1234 & 1235 & 1245 & 1345 & 2345 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Under this projection, the oriented projective point p' is sent to the point $(p' \vee n) \wedge F$, which is calculated as follows.

$$p' \vee n = (p_1, p_2, p_3, p_4, |p|) \vee (0,0,0,1,1)$$

$$= \begin{pmatrix} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \\ 0 & 0 & p_1 & p_1 & 0 & p_2 & p_2 & p_3 & p_3 & p_4 - |p| \end{pmatrix}, \text{ so}$$

$$(p' \vee n) \wedge F = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ p_1 & p_2 & p_3 & 0 & |p| - p_4 \end{pmatrix}.$$

Similarly,

$$(p'' \vee n) \wedge F = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -p_1 & -p_2 & -p_3 & 0 & |p| + p_4 \end{pmatrix}.$$

Dropping the 4^{th} coordinate, which is invariably zero, we obtain two points in $\mathcal{R}^3$ which we call $\gamma_{out}(p), \gamma_{in}(p)$, respectively:

$$\gamma_{out}(p) = (p_1, p_2, p_3, |p| - p_4),$$

$$\gamma_{in}(p) = (p_1, p_2, p_3, -|p| - p_4).$$

Now that we have described this correspondence at the level of points, we can define the *Klein image* of a line L, by

$$\gamma(L) = \{\gamma_{in}(p) |\ p \in L\} \cup \{\gamma_{out}(p) |\ p \in L\}.$$

To simplify our discussion on the main properties of $\gamma(L)$, we introduce some notation. If p is a point in $\mathcal{P}^3$, then the set of all homogeneous coordinates of p, together with the zero vector $(0,0,0,0)$, forms a one-dimensional subspace $H(p)$ of $\mathcal{R}^4$. Similarly, a line L in $\mathcal{P}^3$ gives rise to a two-dimensional subspace $H(L)$ of $\mathcal{R}^4$, defined by

$$H(L) = \bigcup_{p \in L} H(p).$$

Let $C(L)$ the set $\{p' \mid p \in L\} \cup \{p'' \mid p \in L\}$ of corresponding semiprojective points of S^3 that cover the points of L, so

$$C(L) = H(L) \cap S^3.$$

Finally, if the line L does not contain the origin then we can consider the plane of $\mathcal{P}^3$ through the origin and L:

$$\alpha(L) = L \vee o.$$

Theorem 40. *The Klein image $\gamma(L)$ of a line L in $\mathcal{R}^3 \setminus \{o\}$ is a circle in the plane $\alpha(L)$.*

Proof: As $o \notin L$, the subspace $H(L)$ does not contain the "north pole" $n = (0,0,0,1)$ of S^3, the center of the stereographic projection. Consequently, the stereographic image $\gamma(L)$ of $C(L)$ remains a circle [Ev]. A glance at the homogeneous coordinates of $\gamma_{in}(p)$ and $\gamma_{out}(p)$ reveals that both points lie on the line through o and p, and hence lie in $\alpha(L)$. □

Theorem 41. *If A and B are skew lines in $\mathcal{R}^3 \setminus \{o\}$ then $\gamma(A)$ and $\gamma(B)$ are disjoint circles, which are linked exactly once.*

Proof: If $A \cap B = \emptyset$ in $\mathcal{P}^3$ then $H(A) \cap H(B) = \{o\}$ in $\mathcal{R}^4$, and so $C(A) \cap C(B) = \emptyset$. Consequently, the stereographic projections $\gamma(A)$ and $\gamma(B)$ are disjoint as well. We can regard $\gamma(A) \cup \gamma(B)$ as a 2-component link. Next, let R be the line of intersection of the planes $\alpha(A)$ and $\alpha(B)$. Let p_1 and p_2 be the intersections of R with A and B, respectively. From the proof of theorem 40 it is immediate that R intersects $\gamma(A)$ and $\gamma(B)$ in the two Klein images of a, b, $\gamma_{in}(a), \gamma_{out}(a), \gamma_{in}(b), \gamma_{out}(b)$. Since each pair is the stereographic projection of antipodal points, we know that they have reciprocal distances from the origin and that the segment they determine contains the origin [Ev], as in Figure 13. Any two distinct such segments overlap, so the end points of one segment separate the endpoints of the other. $\gamma(A)$ and $\gamma(B)$ are circles in planes through R, having these segments as chords, and hence they are linked. They are linked exactly once because they are geometric circles. □

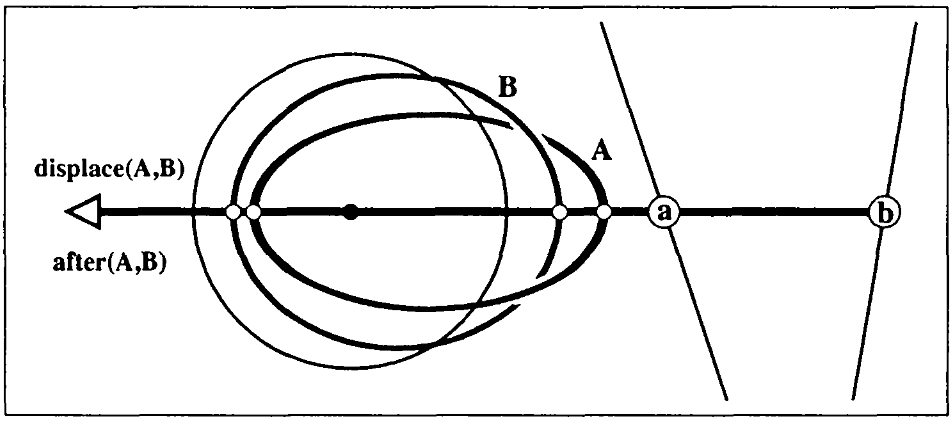


Figure 13. Klein images of skew lines are linked circles.

As we see in Figure 13, for any two lines A, B, the topological link between the circles $\gamma(A), \gamma(B)$ can be specified by an orientation of the directrix $K_{AB} = (o \vee A) \wedge (o \vee B)$, the line of intersection of the planes of the two Klein circles. This orientation should indicate the direction in which the circle $\gamma(A)$ should be translated along the line K_{AB} so that, once it is resized and rotated about the axis K_{AB}, it can coincide with the circle $\gamma(B)$. We call this line, thus oriented, the *Klein displacement* $\mathbf{displace}\,(A, B)$. We are thus free to define

$$\mathbf{displace}\,(A, B) = m_A \vee m_B,$$

where m_A is the midpoint of the segment $[\gamma_{in}(k_a), \gamma_{out}(k_a)]$ and k_a is the point of intersection of the directrix K_{AB} with the line A.

Theorem 42. *If A and B are two lines in $\mathcal{R}^3 \setminus \{o\}$, then the midpoints m_A, m_B of the intervals $[\gamma_{in}(k_a), \gamma_{out}(k_a)]$ and $[\gamma_{in}(k_b), \gamma_{out}(k_b)]$ lie on the directrix $(z \vee A) \wedge (z \vee B)$ and satisfy*

$$\omega(\mathbf{displace}\,(A, B), \mathbf{after}\ _o(A, B)) = +1.$$

Proof: Let $p = k_A$. The midpoint of segment $[\gamma_{in}(p), \gamma_{out}(p)]$ has coordinates $(p_1, p_2, p_3, (p_1^2 + p_2^2 + p_3^2)/p_4^2)$, so it is the reciprocal of the point p in the unit sphere at the origin. Taking the reciprocal relative to the unit sphere is order-reversing on any line through the origin. It follows that the displacement of the Klein circle $\gamma(B)$ relative to the Klein circle $\gamma(A)$ is the same as that of the point k_A relative to k_B. That is, by Theorem 12 of Section 4, it is in the direction $\mathbf{after}\ _o(A, B)$. □

Theorem 43. *For any line L not through the origin, the center c of $\gamma(L)$ is the pole of L with respect to the unit circle in $\alpha(L)$. The radius ρ of $\gamma(L)$ is given by*

$$\rho = \sqrt{1 + ||c||^2}.$$

The circle $\gamma(L)$ is thus the intersection of the plane $\alpha(L)$, whose homogeneous equation is

$$L_{23}x_1 - L_{13}x_2 + L_{12}x_3 = 0,$$

with the sphere whose homogeneous equation is

$$(L_{14}^2 + L_{24}^2 + L_{34}^2)(x_1^2 + x_2^2 + x_3^2 - x_4^2) = \\ 2x_4((-L_{12}L_{24} - L_{13}L_{34})x_1 + (L_{12}L_{14} - L_{23}L_{34})x_2 + (L_{13}L_{14} + L_{23}L_{24})x_3)$$

In particular, if L is a line at infinity then $\gamma(L)$ is the unit circle in the plane $\alpha(L)$.

Proof: We assume that L is not at infinity. In this case, we can consider L as lying in Euclidean 3-space, equipped with the standard norm $||\dots||$ and hence we can define the point q on L such that $oq \perp L$. We will prove that c belongs to oq. The line of intersection of $H(L)$ with the hyperplane $F : x_4 = 0$ contains the $H(d)$ of homogeneous coordinates of the point d at infinity on L. If we identify F with $\mathcal{R}^3$ then $H(d)$ is exactly the line through the origin, parallel to L. Indeed, if we add a fourth coordinate $(= 0)$ to any vector of $H(d) \subset \mathcal{R}^3$ (other than the origin) then we obtain homogeneous coordinates for d. The intersection of $C(L)$ with F leads us to antipodal points $\{d', d''\}$ of S^3 which remain fixed under the stereographic projection. This means that $[d', d''] = [\gamma_{in}(d), \gamma_{out}(d)]$ is a chord of $\gamma(L)$. Because oq is the perpendicular bisector of $[d', d'']$, it must contain the center c of $\gamma(L)$. This already settles the second claim, by considering the right triangle ocz'. That is, since $||z'|| = 1$, Pythagoras' relation gives

$$\rho^2 = 1 + ||c||^2.$$

Furthermore, since the line oq contains the center c of $\gamma(L)$, and intersects $\gamma(L)$ in $\gamma_{in}(q)$ and $\gamma_{out}(q)$, these points must be antipodal in $\gamma(L)$, and bisected by c. Now we will proceed algebraically. Assume that L has projective coordinates

$$L = (L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}).$$

In order to find the (homogeneous) coordinates of q, we first find the meet d of L with the plane I at infinity:

$$d = L \wedge I = (L_{14}, L_{24}, L_{34}, 0),$$

and take the Hodge star M of d within the subspace I:

$$M = (L_{34}, -L_{24}, 0, L_{14}, 0, 0).$$

Next, we join M with the origin, and compute q as the meet with L:

$$q = (M \vee o) \wedge L = (-L_{12}L_{24} - L_{13}L_{34},\ L_{12}L_{14} - L_{23}L_{34},\ L_{13}L_{14} + L_{23}L_{24},\ L_{14}^2 + L_{24}^2 + L_{34}^2).$$

Let

$$\begin{aligned}
q_1 &= -L_{12}L_{24} - L_{13}L_{34} \\
q_2 &= L_{12}L_{14} - L_{23}L_{34} \\
q_3 &= L_{13}L_{14} + L_{23}L_{24} \\
\alpha &= L_{12}^2 + L_{13}^2 + L_{23}^2 \\
\beta &= L_{14}^2 + L_{24}^2 + L_{34}^2
\end{aligned}$$

Compute the squares of the first three coordinates of q:

$$\begin{aligned}
q_1^2 + q_2^2 + q_3^2 &= L_{12}^2L_{24}^2 + 2L_{12}L_{13}L_{24}L_{34} + L_{13}^2L_{34}^2 \\
&\quad + L_{12}^2L_{14}^2 - 2L_{12}L_{14}L_{23}L_{34} + L_{23}^2L_{34}^2 \\
&\quad + L_{13}^2L_{14}^2 + 2L_{13}L_{14}L_{23}L_{34} + L_{23}^2L_{34}^2 \\
&= (L_{12}^2 + L_{13}^2 + L_{23}^2)(L_{14}^2 + L_{24}^2 + L_{34}^2) \\
&= \alpha\beta.
\end{aligned}$$

Here we have made use of the relation

$$2L_{12}L_{13}L_{24}L_{34} - 2L_{12}L_{14}L_{23}L_{34} + 2L_{13}L_{14}L_{23}L_{24} = L_{12}^2L_{34}^2 + L_{13}^2L_{24}^2 + L_{14}^2L_{23}^2,$$

which is obtained by squaring the p-relation

$$L_{12}L_{34} - L_{13}L_{24} + L_{14}L_{23} = 0,$$

which holds for the Plücker coordinates of any line. So the square $|q|^2$ of the $\mathcal{R}^4$ norm of the vector $q = (q_1, q_2, q_3, \beta)$ equals $\alpha\beta + \beta^2$. The square $||q||^2$ of the $\mathcal{R}^3$ norm of the point q equals α/β. In the first part of the proof we have derived that c is the midpoint of the interval bounded by the points $\gamma_{in}(q)$ and $\gamma_{out}(q)$, whose homogeneous coordinates are given by

$$\gamma_{in}(q) = (q_1, q_2, q_3, -|q| - \beta) \text{ and } \gamma_{out}(q) = (q_1, q_2, q_3, |q| - \beta).$$

A preliminary calculation reveals that for any two points of the form

$$u = (q_1, q_2, q_3, s), \quad v = (q_1, q_2, q_3, t),$$

their midpoint has coordinates

$$(q_1, q_2, q_3, \frac{2st}{s+t})$$

and their squared-distance is equal to

$$(q_1^2 + q_2^2 + q_3^2)\left(\frac{1}{s} - \frac{1}{t}\right)^2.$$

Taking $s = -|q| - \beta$, $t = |q| - \beta$, we find

$$\frac{2st}{s+t} = \frac{-2(|q|^2 - \beta^2)}{-2\beta} = \frac{\alpha\beta + \beta^2 - \beta^2}{\beta} = \alpha.$$

So the center of the circle $\gamma(L)$ lies at the point $c = (q_1, q_2, q_3, \alpha)$. Since α and β have the same sign, o does not separate c and q. Computing the distances of q and of c from the origin, we find

$$\begin{aligned} ||c|| = (q_1^2 + q_2^2 + q_3^2)/\alpha^2 &= \alpha\beta/\alpha^2 = \alpha/\beta \\ ||q|| = (q_1^2 + q_2^2 + q_3^2)/\beta^2 &= \alpha\beta/\beta^2 = \beta/\alpha \end{aligned}$$

so q and c are at reciprocal distances, and c is the pole of L relative to the unit circle in $\alpha(L)$.

Now, instead, take $s = \alpha$, $t = |q| - \beta$, in order to compute the squared-radius of the circle:

$$\begin{aligned} &(q_1^2 + q_2^2 + q_3^2)(\tfrac{1}{s} - \tfrac{1}{t})^2 \\ = \ &\alpha\beta(\tfrac{1}{\alpha} - \tfrac{1}{|q|-\beta})^2 = \alpha\beta(\tfrac{1}{\alpha} - \tfrac{|q|+\beta}{\alpha\beta})^2 \\ = \ &(\alpha+\beta)/\alpha \end{aligned}$$

It follows that the circle $\gamma(L)$ has radius

$$\rho = \sqrt{(\alpha+\beta)/\alpha}$$

The sphere with center $c = (c_1, \ldots, c_4)$ and radius ρ has equation

$$c_4^2(x_1^2 + x_2^2 + x_3^2) + (c_1^2 + c_2^2 + c_3^2 - \rho^2 c_4^2)x_4^2 - 2c_4x_4(c_1x_1 + c_2x_2 + c_3x_3) = 0$$

The values just completed for c and ρ produce the equation

$$\alpha^2(x_1^2 + x_2^2 + x_3^2) + (\alpha\beta - \left(\frac{\alpha+\beta}{\alpha}\right)\alpha^2)x_4^2 - 2\alpha x_4(q_1x_1 + q_2x_2 + q_3x_3)) = 0$$

Simplifying and dividing by α, we find

$$\alpha(x_1^2 + x_2^2 + x_3^2 - x_4^2) = 2x_4(q_1x_1 + q_2x_2 + q_3x_3)$$

Substituting the values of the q_i and α, we have the formula as claimed. □

The previous computations enable us to write the coordinates for c explicitly in terms of the line coordinates of L:

$$c = (-L_{12}L_{24} - L_{13}L_{34},\ L_{12}L_{14} - L_{23}L_{34},\ L_{13}L_{14} + L_{23}L_{24},\ L_{12}^2 + L_{13}^2 + L_{23}^2)$$

Remark. From the proof of Theorem 43 it follows that the Klein circle $\gamma(L)$ intersects the unit sphere in two antipodal points q_1 and q_2. Indeed, q_1 and q_2 are the coverings on S^3 of the point at infinity of L and remain fixed under stereographic projection.

Theorem 44. *If L is a configuration of lines which avoid the origin then the link type (isotopy class) of the Klein image $\gamma(L)$ of these lines is an invariant for the isotopy of lines.*

Proof: Let $h = (h_t)_{t\in[0,1]}$ be an isotopy of L. Recall from the introduction that we can view such an isotopy as a continuous map

$$h: [0,1] \to \mathcal{C}$$

with $h(0) = L$. Because the algebraic set of all lines that contain the origin has codimension 2 in $\mathcal{C}$, it cannot disconnect any component of the latter and thence we can always choose h such that $o \notin h_t L$ for any $t \in [0,1]$. This means that the Klein link of $h_t L$ is well defined at any time $t \in [0,1]$. From the previous proof we know that the radius and the (affine) coordinates of the center of $\gamma(L)$ are rational functions of the line coordinates of L. Since the denominator $L_{12}^2 + L_{13}^2 + L_{23}^2$ only becomes zero at the origin, we see that the center and the radius of $\gamma(L)$ vary continuously on $h([0,1])$. Furthermore, the coordinates of the plane $\alpha(L)$ are given by

$$L \vee o = (0, L_{12}, L_{13}, L_{23}),$$

which are clearly continuous in the coordinates L_{ij} of L. Finally, we observe that $\gamma(L)$ is completely determined by $\alpha(L)$, c and ρ, and so $\gamma(h_t L)_{t\in[0,1]}$ is an isotopy of links. □

Isotopy as defined for lines is ***rigid isotopy***, in the sense that it maintains the straightness of the lines in L. We also have a notion of ***flexible isotopy***, namely the topological isotopy of the associated link. Within the framework of this relaxation all knot invariants are at our disposal. Equivalently, two configurations L and L' are flexibly isotopic if and only if there exists an isotopy $(h_t)_{0\le t\le 1}$ of the ambient $\mathcal{P}^3$ such that

$$h_0(L) = L \text{ and } h_1(L) = L'.$$

11. Temari models

For any line configuration L and for any point z not on any of the lines in L, we may construct the central projection of the configuration L to a sphere S with center at the point z. Each line $L \in L$ is mapped to the great circle

$$\sigma_{z,S}(L) \;=\; S \cap (z \vee L).$$

Since the combinatorial structure of the resulting figure of great circles on the sphere does not depend upon the radius of the sphere, we can assume without loss of generality that the sphere S has radius 1, and call the central projection simply σ_z. If the center z is chosen so that it does not lie on any of the $\binom{n}{3}$ ruled surfaces generated by triples of lines in L, then the spherical arrangement $\sigma_z(L)$ will be *regular*, that is, no three of the great circles $\sigma_z(L)$ will share a pair of antipodal points on S. As usual, we call such a point z *regular* with respect to the configuration L.

Let A, B be any two lines in a configuration L. The line **after** $_z(A,B)$ is an orientation of the line $(z \vee A) \wedge (z \vee B)$, and thus passes through the antipodal points of intersection of the great circles $\sigma_z(A), \sigma_z(B)$. This oriented line provides appropriate ***crossing information*** for a ***spherical link diagram*** of the line configuration L. We imagine that the great circle $\sigma_z(B)$ is displaced by a small amount in the direction given by **after** $_z(A,B)$. The circles become linked in space, as in Figure 14.

In Figure 15, we show the spherical link for four lines with "mixed" chiral signature,

123	124	134	234
-1	+1	-1	+1

In the Japanese folk tradition, one finds such models constructed on solid balls with brightly coloured strings. We borrow the Japanese term, and call $\sigma_z(L)$ a *temari* model of the configuration L.

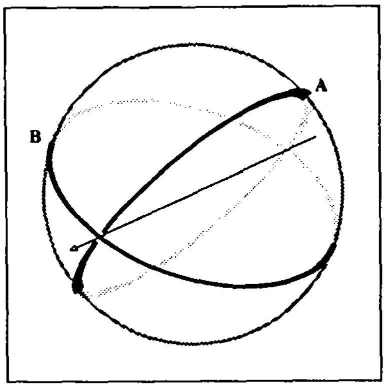


Figure 14. The oriented line after $_z(A, B)$ reveals the linking of the corresponding circles in the spherical model.

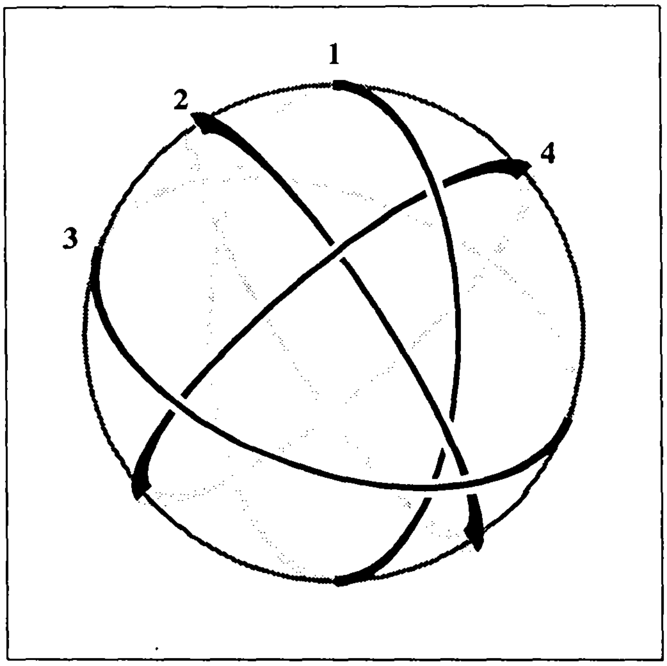


Figure 15. The Temari model of a configuration of four lines.

We can represent spherical link diagrams in the plane by stereographic projection of the temari model. Suppose that the given spherical diagram is regular, so not any point of S belongs to more than two circles of the diagram. Let f be a *face* of the arrangement of great circles, that is, f is a connected component of the complement of the circles relative to S. Let φ be the stereographic projection from a point $p \in f$ into the plane Q tangent to S in the antipode of p. This projection maps the arrangement of great circles $\sigma_z(A)$ onto n circles φ_A in the plane Q in such a way that each pair of circles φ_A, φ_B intersect exactly twice, at the images $\varphi(c), \varphi(d)$ of the antipodal points c, d of intersection of σ_A, σ_B. Say the oriented line after $_z(A, B)$ *enters* the sphere at c and *exits* at d. We then choose the crossings

$$\text{under}\,(\varphi_A, \varphi_B) \;=\; \begin{cases} +1 & \text{at } \varphi(c), \\ -1 & \text{at } \varphi(d). \end{cases}$$

Here we assume the natural extension of the notion of under to crossing of curves, from that of lines. Such an image is constructed in Figure 16 for the temari model of Figure 15. The selected face is front and center in Figure 15. The back hemi-sphere of the temari model has stereographic image within the gray disc of Figure 16.

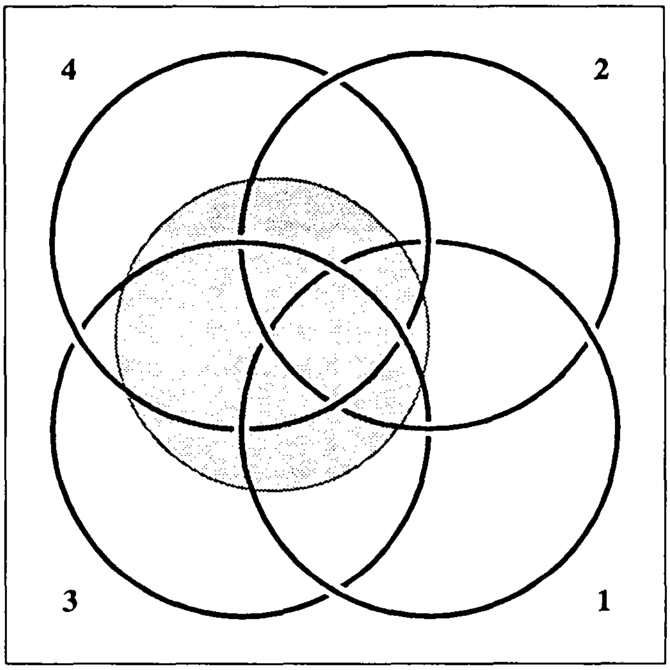


Figure 16. Stereographic projection of the Temari model of Figure 15.

Theorem 45. *Let $L \in C$ be a configuration such that the origin does not belong to any of the lines of L. Then the temari model for L around o is a spherical diagram for the Klein link $\gamma(L)$.*

Proof: By Theorem 42, the relative displacement of circles $\gamma(A), \gamma(B)$ along the directrix $K = (o \vee A) \wedge (o \vee B)$ agrees with their displacement in the temari model, where it is given by the ray $\text{after}_o(A, B)$. □

Recall that the planar layout of a line configuration L can be viewed as the isotopy limit of its projection, and that, during this isotopy the planar layout remains fixed. The same phenomenon can be proved for the temari model, which, under the conditions of Theorem 45, is a constant spherical diagram during a limit isotopy of Klein links.

Theorem 46. *Let L be a line configuration with no line through the origin. There exists an isotopy of lines, $\{L_t; 1 \geq t > 0\}$, such that*

1. $L_1 = L$;
2. *For each $0 < t \leq 1$, the lines $L_i(t)$ of L_t lie in the planes $\alpha(L_i)$, avoiding o;*
3. *The limit $\lim_{t \to 0} L_t$ in $(G_{2,4})^n$ is exactly the (planar) configuration $(I_1, \ldots, I_n)$ of the lines of infinity of the planes $\alpha(L_i)$.*

If L is a configuration in $\mathcal{R}^3$ that satisfies the conditions of Theorem 45, then it does so also at any time $t > 0$, and the spherical diagram of $\gamma(L_t)$ around z, the temari model of L_t) is constant with respect to t.

Proof: For any line $L \in L$ we define

$$L(t) = (L_{12}, L_{13}, tL_{14}, L_{23}, tL_{24}, tL_{34}).$$

Since

$$L_i(t) \vee L_j(t) = t(L_i \vee L_j),$$

the configuration L_t is a configuration of skew lines for $t \neq 0$. Further, we observe that

$$\alpha(L_i) \;=\; o \vee L_i(t) \;=\; (0, L_{12}, L_{13}, L_{23})$$

is an invariant for this isotopy, and so we conclude that $o \notin L_i(t)$ for all $t \neq 0$. Finally,

$$\lim_{t \to 0} L_i(t) = (L_{12}, L_{13}, 0, L_{13}, 0, 0),$$

which is the line at infinity I_i of $\alpha(L_i)$. □

12. Transformation of a Line Diagram to a Link Diagram

Suppose we are given a planar layout $D = (H, \mathbf{under})$ as a representation of some configuration L, and we wish to construct a diagram for the Klein link $\gamma(L)$. We know that this diagram is well-defined up to link isotopy, independent of the planar layout of L, by Theorem 44. It is the purpose of this section to provide a direct combinatorial construction of such a link diagram, starting from a given planar layout.

Recall that the definition of planar layout includes the assumption that the projection is proper: no line in L lies in the plane at infinity, and no pair of lines in L become parallel in projection. For the present construction we assume also that the planar layout is regular: no triple of lines in L are concurrent in projection.

As a first step, we modify the planar layout to produce a geometric braid. This involves the selection of an oriented line B not parallel to any line in D, which we call the *base* of the braid. Choose a circular disc C which contains all the crossing points of D, and let D_C be the restriction of the diagram D to that disc. We can find a self-homeomorphism of the disc C such that the segments of lines H_i of D on C become pseudolines meeting the circumference of C in pairs of antipodal points, and such that the straight line segment through those two points has the same slope as the line H_i. Now *reverse all the crossings* in order to pass from the planar layout to a portion of the corresponding link diagram.

See Figure 17. Here, we choose a planar layout which will yield the temari model of Figure 15, and the link diagram of Figure 16. The base is taken to be the positively oriented x-axis. Counter-clockwise rotation of the base brings it in line with $b_1, \ldots, b_4, a_1, \ldots, a_4$ in that order.

The base B of the braid provides an additional pair of antipodal points on the circumference, and a separation of the extremities of the pseudolines into two bundles of n points each[10]. Without loss of generality, we relabel the pseudolines and their end points a_i, b_i so that the b_i occur in increasing order, counter-clockwise between the antipodal points of the base B. The same is automatically true for the opposite end points a_i.

Now deform the disc C with its pseudoline diagram, until it fits in the rectangle of height h, width $n+1$, with lower-left corner at the origin, and with the points b_i at $(n+i-1, h)$, and the points a_i at $(i, 0)$. The resulting pseudoline diagram, together with its crossing information and the base, constitutes a geometric braid that we call α.

[10] Another convenient choice for the base would be the positive y axis. In that case, the lines become numbered in increasing order of their slopes.

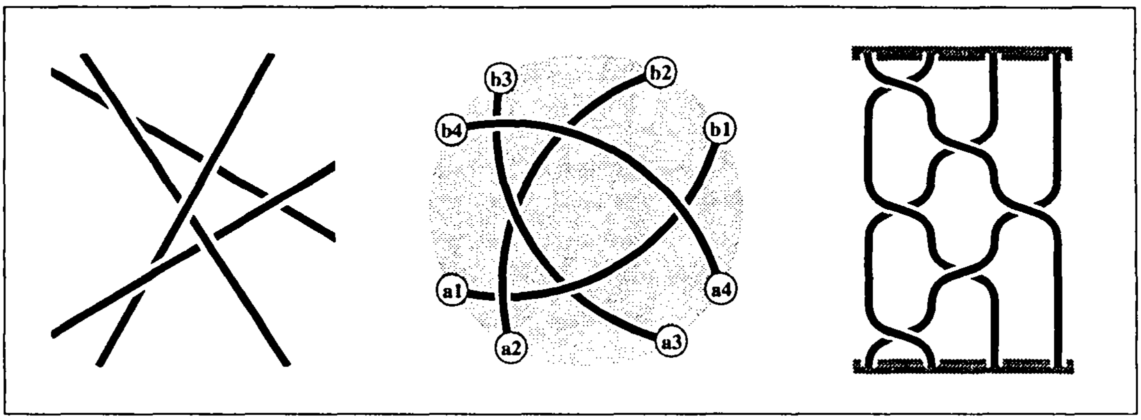


Figure 17. The braid construction: first stages

Let $\hat{\alpha}$ be the geometric braid obtained by rotating the plane by a half turn in 3-space about the vertical line with equation $2x = n + 1$, then translating the plane h units upwards. This produces a glide reflection along a vertical axis. The composite $\beta = \alpha\hat{\alpha}$ of these braids can be *closed* to form an n-component link by joining the points a_i of α to the points b'_i of $\hat{\alpha}$, respectively, by n disjoint strands lying entirely outside the rectangle R now occupied by the braid β. There are of course several ways to "part" these added strands, letting them pass left or right the rectangle R, but it is clear that the different choices all lead to isotopic links. We call the resulting link $L(D)$. We will prove that $L(D)$ has the same link type as the Klein link $\gamma(L)$, where L is any lifting of D. Notice that this result immediately implies that the link type of $L(D)$ is independent of the chosen base for D.

On the left side of Figure 18, we show the resulting link diagram for the example of Figure 17. On the right side, we show an alternate representation of the same link. Here, we transform the braid α by a glide reflection along a horizontal axis. These two images should be compared with the stereographic image of the temari model shown in Figure 16.

Let L be any lifting of D, with respect to an arbitrary projection $\pi_{z,Q}$. The isotopy type of L does not depend on a particular choice of lifting or projection, so, by Theorem 44, the choice of a lifting does not affect the isotopy type of the Klein link $\gamma(L)$. For the same reason, we may assume that the center z lies at the origin o, since a change of origin corresponds to a translation and hence an isotopy of the ambient space.

Theorem 47. *Let L be a configuration with planar layout $D = (H, \text{under})$. The Klein link has the same isotopy type as the link diagram $L(D)$.*

Proof: Let z denote once again the center of projection. Without affecting the link type of $\gamma(L)$, we may assume that z is the origin of $\mathcal{R}^3$ and that it does not belong to any line of L. The temari model, the Klein link and the planar layout are all well-defined for L with respect to z. By Theorem 45, we know that the temari model is a spherical diagram for the Klein link. So, it is sufficient to show that $L(D)$ is a planar diagram representing this spherical diagram. To this end, we consider the part of the temari model facing to the screen of projection Q. Take a plane Q_z through z and parallel to Q, and consider the corresponding hemisphere S^+ of S between Q and Q_z. The arrangement of circles underlying the temari model induce a cell decomposition of S^+ which is isomorphic to the cell complex induced by H of a disc that contains all crossings of D in its interior. (Figure 16.)

Take two lines A, B in L. Assume that the directrix $K = \text{after}_z(A, B)$ exits the sphere S at a

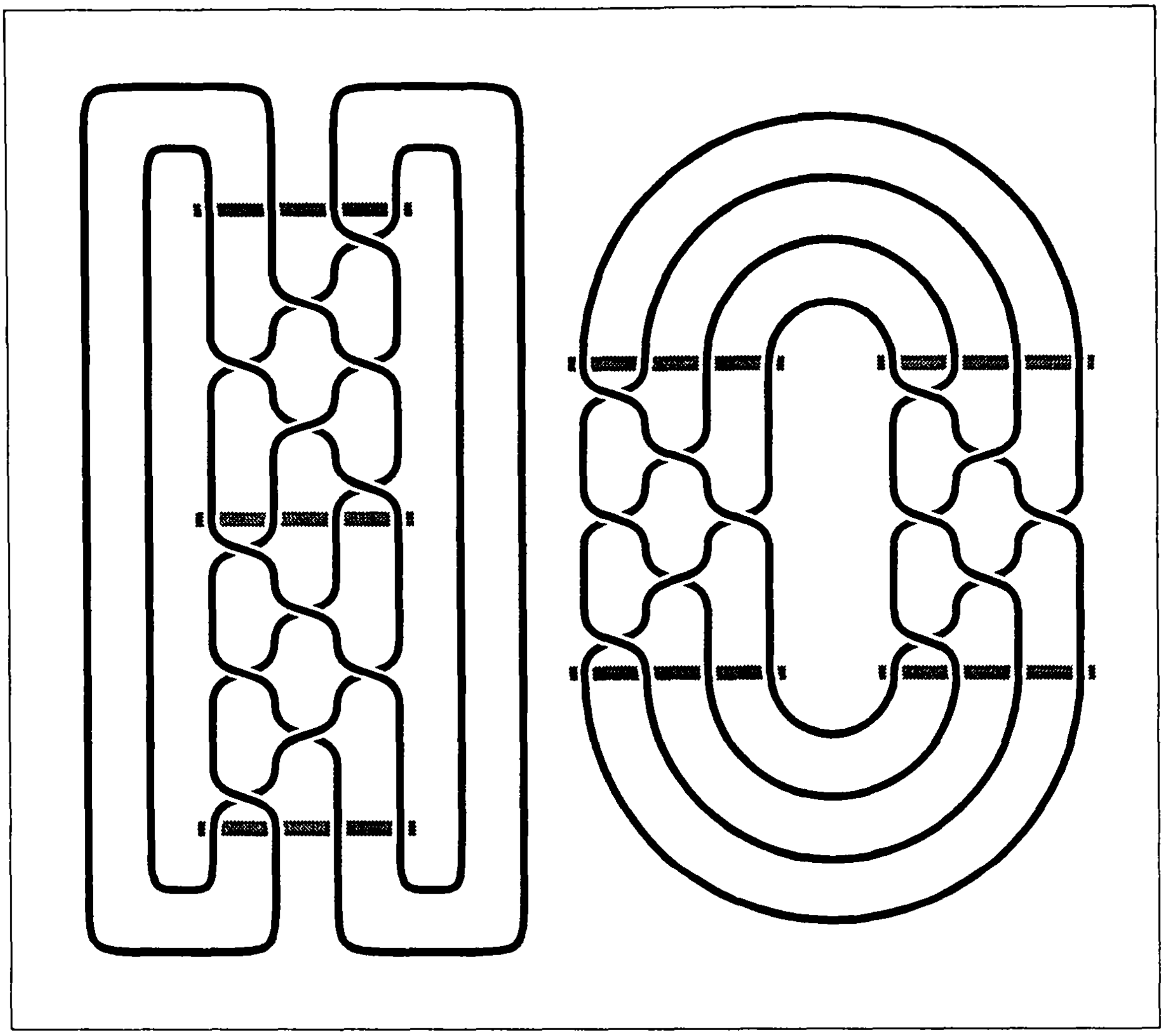


Figure 18. Two images of the geometric braid for the configuration in Figure 17.

point on the hemisphere S^+ facing the plane Q. At the crossing point $K \wedge Q$ in the planar layout D, $\mathbf{under}\ (A, B) = +1$. In the corresponding braid α of the link diagram $L(D)$, this crossing is reversed, so $\mathbf{under}\ (c(A), c(B)) = -1$, which agrees with the displacement

$$\mathbf{displace}\ (A, B) \ \sim\ \mathbf{after}\ _z(A, B) \ \sim\ K,$$

the circle $\gamma(B)$ being closer to the plane Q than $\gamma(A)$ in the temari model. □

13. The hierarchy of isotopy classifications

This is an appropriate moment to compare the different kinds of relaxation of rigid isotopy of lines, encountered in the previous sections. Recall that pseudoline diagrams D_0, D_1 are *equivalent* if they are connected by a finite series of diagram moves. If the diagrams D_i are in fact line diagrams, and all the intermediate situations are stretchable, then the diagrams are *rigidly equivalent*. If they can be joined by a *continuous* family of line diagrams possessing a finite set of isolated $||$ - and $\not\times$ -singularities, they are *homotopic.* If the line diagrams and all intermediate situations of the isotopy are liftable to configurations of skew lines in space, the diagrams are ***lift-homotopic.*** The following theorem summarizes the known implications which link these concepts.

Theorem 48. *Let L_1 and L_2 be two proper line configurations with respect to the projection $\pi_{z,Q}$, and let D_1 and D_2, respectively, be their planar layouts. Then the following implications hold:*

$$\begin{array}{cll} & (1) & L_1 \text{ and } L_2 \text{ are isotopic line configurations} \\ \Leftrightarrow & (2) & D_1 \text{ and } D_2 \text{ are lift-homotopic line diagrams} \\ \Rightarrow & (3) & D_1 \text{ and } D_2 \text{ are homotopic line diagrams} \\ \Rightarrow & (4) & D_1 \text{ and } D_2 \text{ are rigidly equivalent line diagrams} \\ \Rightarrow & (5) & D_1 \text{ and } D_2 \text{ are equivalent pseudoline diagrams} \\ \Rightarrow & (6) & \gamma(L_1) = L(D_1) \text{ and } \gamma(L_2) = L(D_2) \text{ are isotopic links} \\ \Rightarrow & (7) & L_1 \text{ and } L_2 \text{ have the same chiralities} \end{array}$$

Proof: The first equivalence ($1 \Leftrightarrow 2$) is the main theorem of Section 9. The implications $2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$ follow immediately from the definitions.

($5 \Rightarrow 6$) We first observe that a $||$-move on a diagram D does not change the link $L(D)$. After the move, each pseudoline of the modified diagram D' meets the other pseudolines in the same circular order as before. With an appropriate choice of base for the construction of the braid $\alpha(D)$, we can arrange that the crossing c involved in the $||$-move occurs at the beginning of the braid. The $||$-move shifts this crossing c to the opposite crossing c' at the end of the braid α', where it produces in $\hat{\alpha}'$ a crossing $\hat{c}'$ equal to c. We have simply shifted the base k line of the link $L(D)$.

If two pseudoline diagrams differ by one $\not\times$-move then the associated link diagrams can be connected by two Reidemeister-3 moves, one in the α-part and one in the $\hat{\alpha}$-part. By Reidemeister's theorem we can conclude that the associated Klein links are isotopic.

($6 \Rightarrow 7$) Suppose that the Klein links $\gamma(L_0)$ and $\gamma(L_1)$ are isotopic. Choose an arbitrary orientation of each component of the link $\gamma(L_0)$ and, following the isotopy, find the corresponding orientation of the components of $\gamma(L_1)$. For each pair of lines $A, B \in L$,

$$lk(\gamma(A), \gamma(B)) \;=\; \textbf{link}\,(A, B) \;=\; \textbf{sign}\,[AB]$$

by Theorem 42. (See Figure 5.) It follows that the chiral signature of any line configuration is determined by the link type of its Klein link. Since the linking number of each pair $\gamma(A), \gamma(B)$ of components of the Klein link is preserved under link isotopy, it follows that the chiral signatures of L_0 and L_1 are equal. □

14. The Kauffman Polynomial

L.H.Kauffman invented a polynomial for link diagrams by making use of a "state model" [Ka]. An appropriate specialization of the variables in this polynomial yields the Jones polynomial [Jo]. By considering a slightly modified state model we will intoduce a "Kauffman polynomial" for line diagrams, a substitution of which gives us an invariant for diagram equivalence[11].

Let $D = (H, \textbf{under})$ be a diagram of n pseudolines. As in the procedure that associates the α-braid to D, we choose a closed circular disc C containing all crossings of D, and replace H by an isomorphic arrangement of pseudolines such that each pseudoline H_i intersects the boundary of C in a pair of antipodal

[11] Essentially the same polynomial has been considered by Drobotukhina [Dr] for links in projective 3-space, and applied to configurations of lines by Mazurovskii [Ma]

points. Let S_i denote the segment of H_i that lies in the interior of C. The configuration $S = S_1 \cup \ldots \cup S_n$ can be modified locally at some crossing $S_i \cap S_j$ by performing two possible splits, distinguished by the crossing function under as follows. If the upper line at any given crossing is rotated CCW, it sweeps out a pair of regions which we call the A regions (see Figure 19); the complementary regions we call B. If both lines are broken at the crossing, they can be rejoined so as to unite the A or the B regions. We call these operations the A and B *splits*, respectively.

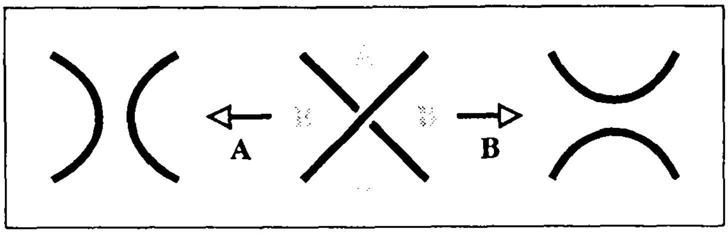


Figure 19. Splits A and B of a crossing.

After one such split we still have a collection of n curves in C, but now the number of intersections is diminished by one. After $\binom{n}{2}$ splits, the diagram is reduced to a disjoint union of n simple closed curves in the projective plane. A *state* Σ is any collection of curves resulting from an arbitrary choice of $\binom{n}{2}$ splits in S. We put $a(\Sigma)$ and $b(\Sigma)$ to be the number of A-splits and B-splits, respectively, which transform S to Σ. Notice that $a(\Sigma) + b(\Sigma) = \binom{n}{2}$. Up to this point we have merely followed the definitions and notations of Kauffman for link diagrams.

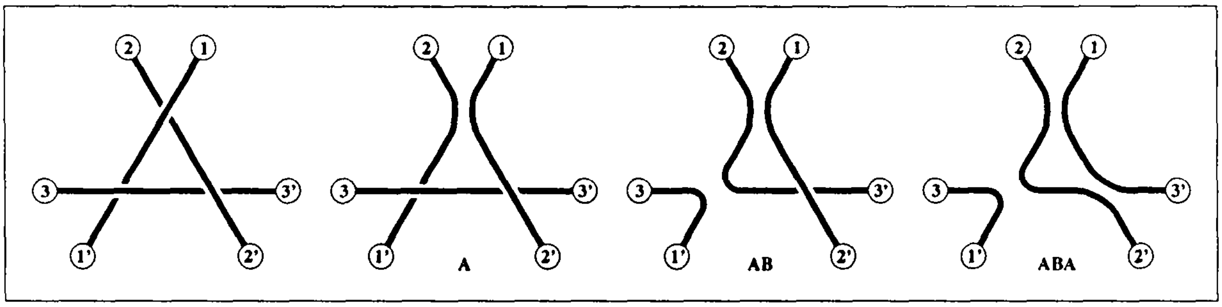


Figure 20. Splits leading to the ABA state for three lines.

Next, we consider the canonical map π from the closed disc C to the projective plane $\mathcal{P}^2$. This map

$$\pi : C \to C/\sim$$

identifies two points $x \sim y$ if and only if $x = y$ or x and y are antipodal points on the boundary of C. Now we can define $|\Sigma|$ as the number of disjoint curves (connected components) of $\pi(\Sigma)$. Finally, we define the *Kauffman bracket* $[D]$ of the diagram D by

$$[D] = \sum A^{a(\Sigma)} B^{b(\Sigma)} d^{|\Sigma|-1} \in \mathcal{Z}[A, B, d],$$

where we sum over all $2^{\binom{n}{2}}$ possible states for S.

We can also consider a *partial state* Σ', obtained from S by splitting only *some* of the crossings. For a partial state Σ' we can define complete states as well, obtained by splitting the remaining crossings. Consequently, the Kauffman bracket $[\Sigma']$ also makes sense for partial states. This enables us to state the following *recursion formula*.

Theorem 49. *Fix some crossing of a diagram D and let D_A (resp., D_B) denote the partial state that is obtained by the A-split (resp., B-split) of that crossing. Then*

$$[D] = A[D_A] + B[D_B].$$

□

In Figure 21 we compute the Kauffman bracket $[D]$ for a diagram D with three lines. The A^3 state has one component, the three A^2B states have two components, the three AB^2 states have one component, and the B^3 state has two components. Thus the Kauffman bracket polynomial is

$$A^3 + 3A^2Bd + 3AB^2 + B^3d.$$

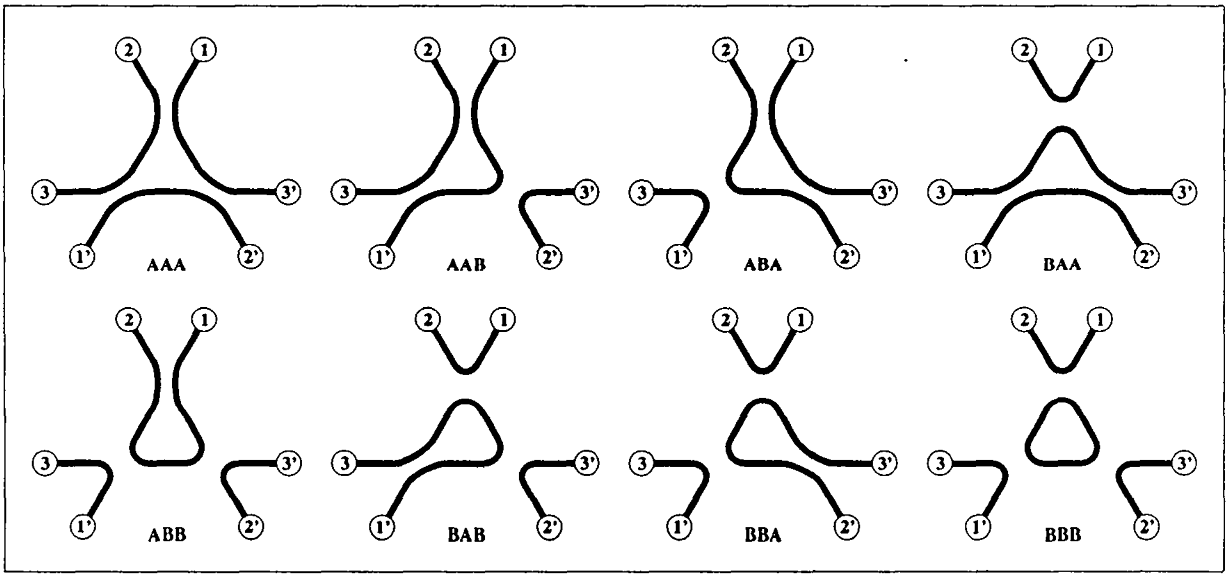


Figure 21. The eight states for three lines forming a positive vortex.

Lemma 50. *If D and D' are two diagrams which differ by a move of type $||$, then $[D] = [D']$.*

Proof: Let c be the crossing involved in a $||$-move. The A-splits of D and D' at c are equivalent under an orientation-preserving homeomorphism of the oriented projective plane, as are the B-splits (see Figure 22.). By Theorem 49, the Kauffman brackets $[D]$ and $[D']$ are equal. □

Unfortunately, the Kauffman bracket is not invariant under the $\times$-move. To overcome this, we consider the ring homomorphism:

$$\begin{array}{rcll} \psi : \mathcal{Z}[A,B,d] & \to & \mathcal{Z}[A,A^{-1},B,B^{-1}], & \text{by substituting} \\ A & \to & A & \\ B & \to & B & \\ d & \to & -A^{-1}B^{-1}(A^2+B^2) = -AB^{-1} - A^{-1}B & \end{array}$$

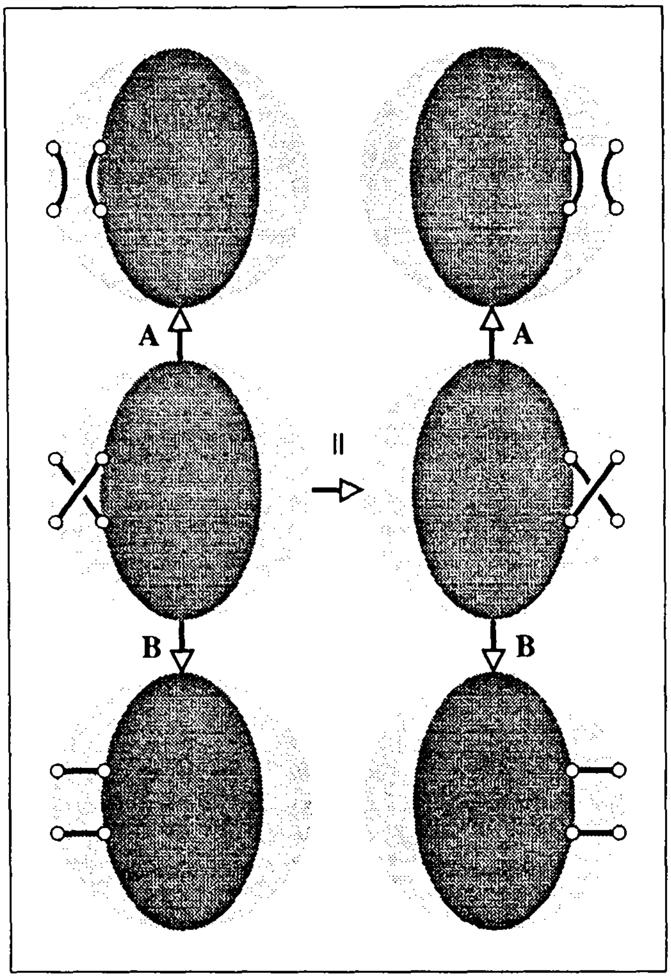


Figure 22. The A- and B- splits of a crossing altered by a $||$ -move.

We define the *Kauffman polynomial* of a (pseudo)line diagram by $< D >= \psi([D])$.

This specialization of the Kauffman bracket polynomial is far from innocent. The coefficients of the Kauffman bracket for planar layouts modified by a $\times$-move can be radically different. For instance, the diagrams D and D' in Figure 23 are both configurations with chiral signature all positive. They have brackets

$$\begin{gathered} A^6(d) \; + \; A^5B(4+2d^2) \; + \; A^4B^2(4+10d+d^3) \; + \; A^3B^3(8+8d+4d^2) \; + \\ A^2B^4(8+6d+d^2) \; + \; AB^5(4+2d) \; + \; B^6(d) \; + \end{gathered}$$

and

$$\begin{gathered} A^6(1) \; + \; A^5B(3+3d) \; + \; A^4B^2(6+6d+3d^2) \; + \; A^3B^3(6+12d+d^2+d^3) \; + \\ A^2B^4(9+3d+3d^2) \; + \; AB^5(3+3d) \; + \; B^6(d) \; + \end{gathered}$$

respectively.

The passage from D to D' is accomplished by a series of three $||$ -moves and three $\times$-moves, that is, by the rotation of the gray line by 180° about the indicated point as center. The diagrams D, D' have the same Kauffman polynomial,

$$\begin{aligned} & - \; 2A^5B \; - \; 3A^4B^2 \; - \; A^3B^3 \; - \; B^6 \; - \; A^{-1}B^7 \\ = & \; (\; - \; 2A^3B \; + \; A^2B^2 \; - \; AB^3 \; + \; B^4 \; - \; A^{-1}B^5) \; (A+B)^2 \end{aligned}$$

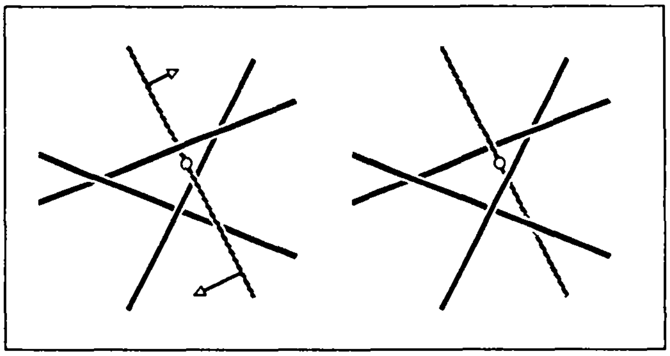


Figure 23. Configurations with the same chiral signature, differing by the rotation of a shellable line.

Theorem 51. *If D and D^* are mirror images (obtained from one another by reversing all crossings), then*

$$< D^* > (A,B) \;=\; < D > (B,A)$$

Proof: Each A-split for D is a B-split for D^* and vice versa. Consequently, $[D^*]$ is obtained from $[D]$ by exchanging A and B. Since the substitution $d = -AB^{-1} - A^{-1}B$ is symmetric in A and B, the same exchange of A and B maps $< D >$ to $< D^* >$. □

Example. For the diagram of Figures 20 and 21, three lines with positive chirality, we compute

$$< D >= -2A^3 - AB^2 - A^{-1}B^4.$$

If we instead take three lines with negative chirality, then

$$< D >= -A^4B^{-1} - A^2B - 2B^3.$$

Lemma 52. *If D and D' are two pseudoline diagrams which differ by a move of type $\ast$ then*

$$< D >=< D' > .$$

Proof: Let H_1, H_2 and H_3 be the three lines involved in the $\ast$-move. Suppose that under $(H_1, H_2) =$ under $(H_1, H_3) = -1$ and under $(H_2, H_3) = 1$. The diagrams D and D' differ only in a disc containing the three crossings c_{12}, c_{13}, c_{23}, and in the manner indicated in Figure 24.

For both D and D' we compute eight partial states. The idea is to pair these off insofar as possible, so that corresponding partial states have the same bracket polynomial. When this is no longer possible, we partition the remaining partial states in such a way that the sum of the bracket polynomials for the states in any block of the partition will have a factor of $A^2 + dAB + B^2$, a factor which will vanish under the ring homomorphism φ transforming bracket polynomials to Kauffman polynomials. The required correspondence is shown in detail in Figure 25.

As indicated in the left-most column of Figure 25, the bracket polynomial $[D]$ has a sum of three terms equal to

$$A\ (A^2 + dAB + B^2)\ [12|31'|2'3'],$$

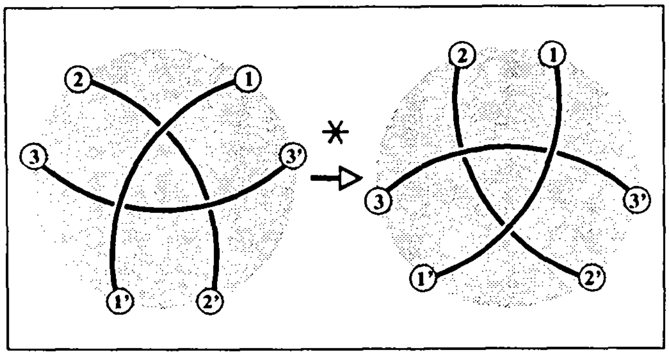


Figure 24. Local modification of a pseudoline diagram during a $\ast$-move

where $[12|31'|2'3']$ denotes the bracket polynomial of the partial state obtained by joining vertex 1 to vertex 2, vertex 3 to vertex $1'$, vertex $2'$ to vertex $3'$. The factor d occurs in the second of these terms because of the extra loop in the corresponding partial state. It is not necessary to specify whether this partial state arises from D or from D', because these two configurations agree aoutside the disc containing the crossings $c_{12}.c_{13}, c_{23}$. Similarly, $[D']$ has a sum of three terms equal to

$$A\ (A^2 + dAB + B^2)\ [23|1'2'|3'1].$$

Both of these expressions become zero under the ring homomorphism φ. The remaining five pairs of terms match, as shown, in five rows of Figure 25, so the Kauffman polynomials $< D >, < D' >$ are equal. □

If we combine Theorems 50 and 52, we obtain:

Theorem 53. *The Kauffman polynomial is an invariant for the equivalence of pseudoline diagrams.* □

In Figure 26, we show the three pairs of unlabelled configurations of six lines having the same chiral signature, but which can be distinguished by their Kauffman polynomials. The first two pairs of configurations are opposite (obtained by reversing "over" and "under" at all crossings). On the top of these two columns is a stacked configuration; on the bottom, a configuration not so representable. The latter are isotopic to the figures "L" and "L'" in [Ma]. The final pair of configurations (on the right) is self-opposite in the sense that one can be obtained from the other by reversal of "over" and "under", followed by a permutation of the lines. They are isotopic to the configurations "M" and "M'" in [Ma].

In the first two examples of Figure 26 we have arranged the two diagrams in such a way as to show how one can pass from one to the other by an illegal "vortex move" (see Figure 27). In a *vortex move*, or ⊚-move, a local disc carrying three crossings of a vortex is excised, rotated by 180°, and replaced. Such a move preserves the chiral signature, but can be used to pass from one isotopy class to another. The proof of the following theorem will be found in [Pe_4].

Theorem 54. *If D and D' are diagrams of configurations having the same chiral signature, then they can be transformed to one another by a finite series of moves of three types: $||$, $\ast$, ⊚.* □

15. Spindles

As candidates for "canonical forms" for at least some isotopy classes of families of lines, we need examples of families which are easy to visualize, and whose chiral signatures are easy to calculate. We

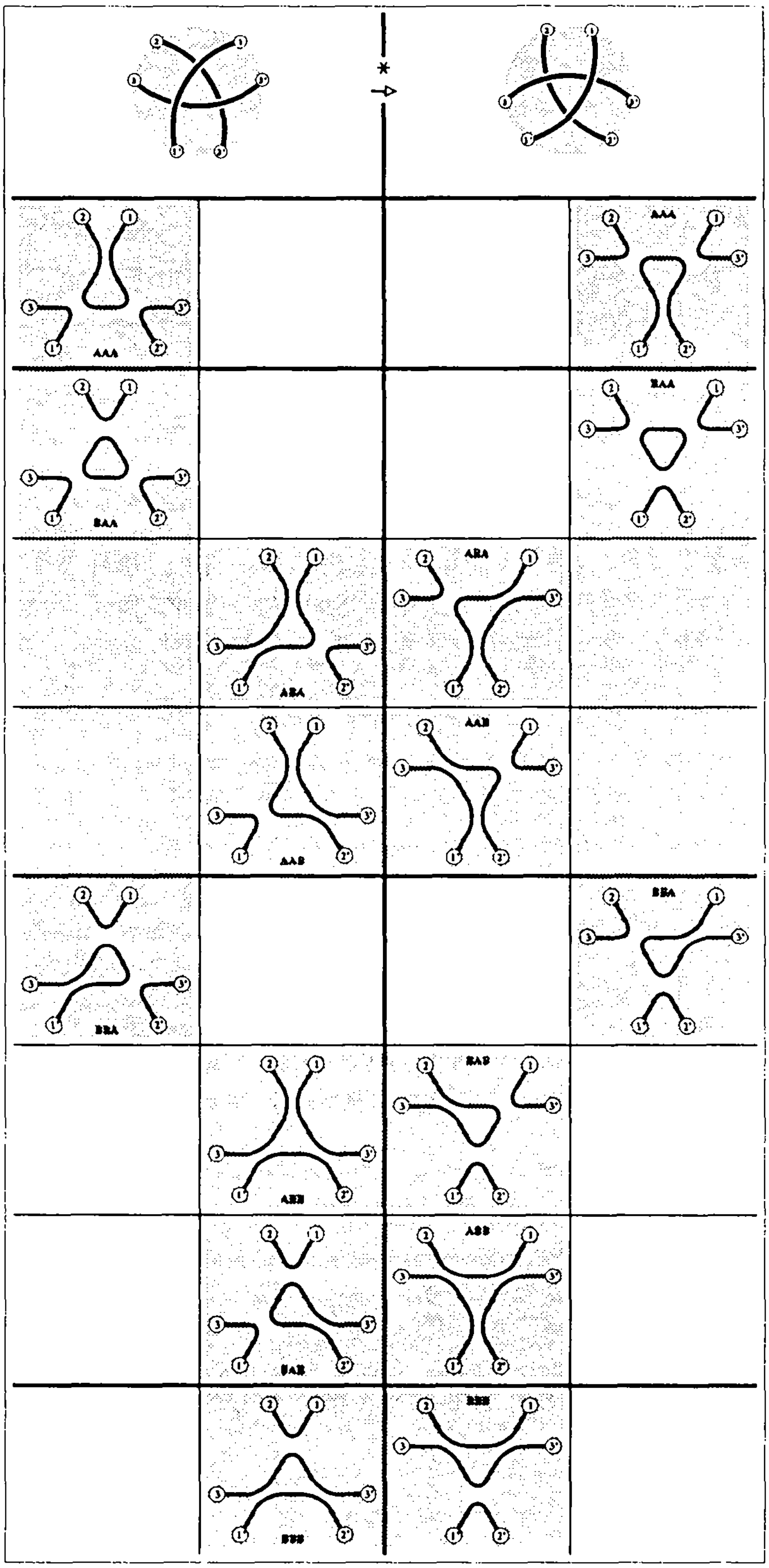


Figure 25. The cancellation of partial states in a ⚹-move

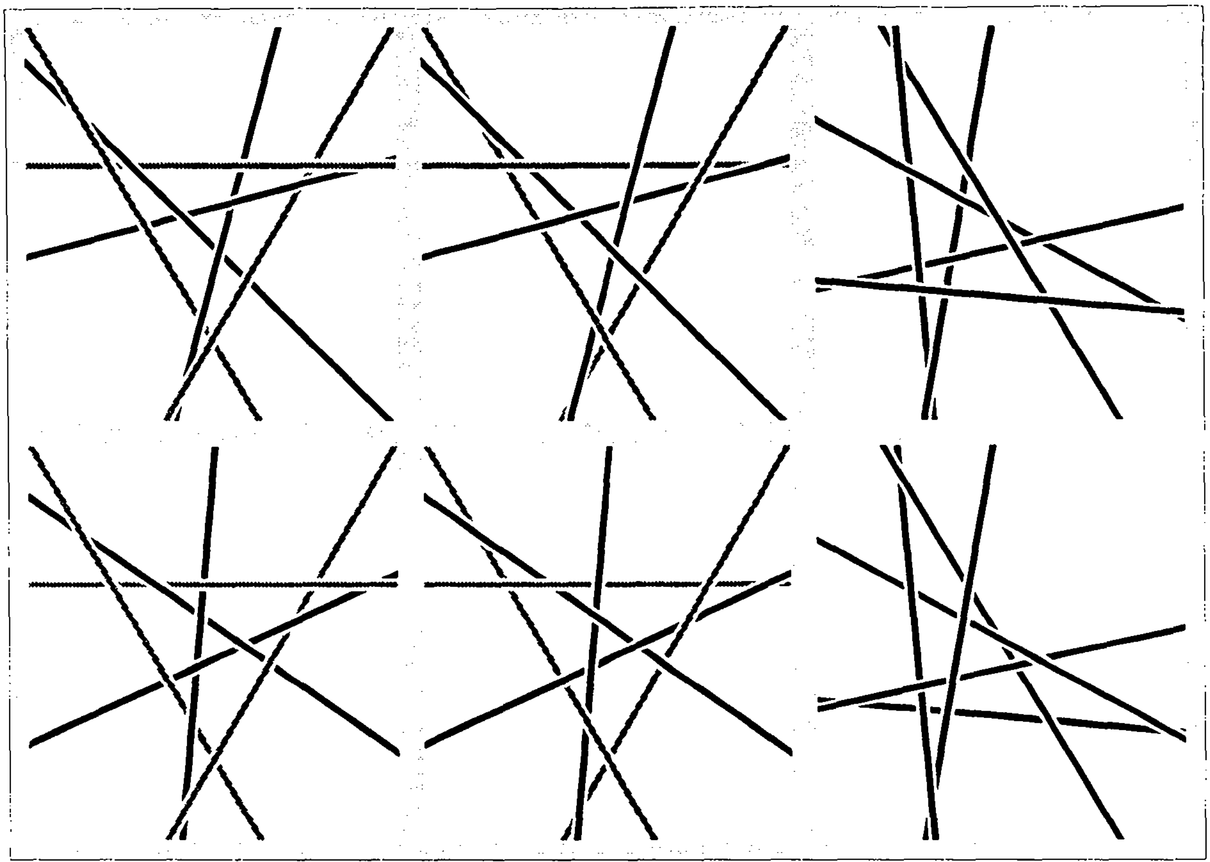


Figure 26. Three pairs of configurations distinguished by their Kauffman polynomials, but not by chiral signature.

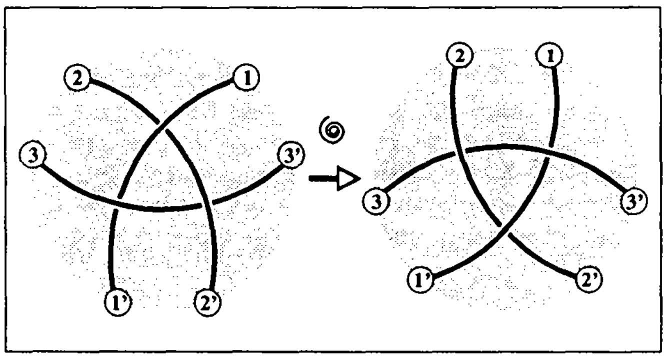


Figure 27. The illegal "vortex-move", which preserves chiral signature.

consider all families of lines which meet two fixed skew lines, or *directrices*[12]. An isotopy will move any such family to a position in which the directrices are the horizontal line at infinity, "H", and the z-axis, "Z". The lines of the family are then parallel to the xy-plane, intersect Z at distinct heights, and point in distinct directions. We call such a configuration a *spindle*, and if it contains n lines, an *n-spindle*. Each line A in the family intersects the plane $y = -1$ at a point whose x-coordinate we call $\kappa(A)$, and

[12] The Russian school calls these "join configurations".

intersects the z-axis at a height which we call $\lambda(A)$. Note that we can equally well describe $\kappa(A)$ as the first coordinate of the vector $(\kappa(A), -1, 0, 0)$, which coordinatizes the point of intersection of the lines A and H.

Theorem 55. *Every n-spindle is isotopic to a spindle in which the values $\kappa(A)$ and $\lambda(A)$, for lines A in the family, are two linear orderings of the numbers $1, \ldots, n$.*

Proof: Any two families of 2 lines in projective space are isotopic. The entire space can simply be rotated in such a way as to move the first line into any desired position. Leaving this first line in place, the second line can be moved freely around it, to take up any desired position skew to the first line. Thus we are free to move any given spindle until its directrices coincide with the lines H and Z. This motion of the directrices cannot cause two lines of the spindle to intersect, because lines which meet two skew directrices can only intersect each other at a point on one of those two directrices. For this same reason, we can isotopically move any single line of the spindle by changing its point of contact with one of the directrices, as long as we don't pass either of its "neighbors" on that directrix. In this way, the points of intersection with the directrices can be successively shifted until they fall precisely at the first n integer points on the lines H and Z. $\square$

Henceforth, we shall use the terms ***spindle*** and ***n-spindle*** to refer only to those of the special nature specified in Theorem 55, that is, with directrices H and Z, meeting H and Z at their first n integer points[13].

From any n-spindle, we obtain two linear orders on the family of lines. By abuse of notation, we call those linear orders simply κ and λ:

$$\kappa, \text{ given by } (\kappa^{-1}(1), \ldots, \kappa^{-1}(n)), \text{ and}$$

$$\lambda, \text{ given by } (\lambda^{-1}(1), \ldots, \lambda^{-1}(n)),$$

and refer to the spindle as the spindle (κ, λ). The linear orders κ and λ determine the chiral signature of a spindle, as follows.

Theorem 56. *In a spindle, each triple A, B, C of lines has chirality*

$$\chi_{ABC} = (\kappa_A - \kappa_B)\ (\lambda_A - \lambda_B)\ (\kappa_A - \kappa_C)\ (\lambda_A - \lambda_C)\ (\kappa_B - \kappa_C)\ (\lambda_B - \lambda_C)\ ,$$

a quantity which is positive (resp., negative) when passage from order κ to order λ is an even (resp., odd) permutation of the triple of lines.

Proof: A line A in the spindle has coordinates

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \kappa(A) & -1 & 0 & 0 \end{pmatrix} \vee \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & \lambda(A) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 0 & \kappa(A)\lambda(A) & \kappa(A) & -\lambda(A) & -1 & 0 \end{pmatrix}.$$

[13] One can think of a spindle as a set of horizontal bars nailed at different heights to a single vertical support. Constructions of this sort are still employed in some parts of the world as "echelles de poule", to enable chickens to climb back up to their roost when they have finished feeding on the ground. Our somewhat more general spindles are more practical for use by chickens when they are monotone, or, as we shall see, when they have chiral signature $(+1, \ldots, +1)$ or $(-1, \ldots, -1)$.

Thus, for any pair A, B of lines in the spindle,

$$A \vee B = \kappa_A \lambda_A - \kappa_A \lambda_B - \kappa_B \lambda_A + \kappa_B \lambda_B$$

$$= (\kappa_A - \kappa_B)\ (\lambda_A - \lambda_B),$$

and for any triple A, B, C in the spindle, we find χ_{ABC} given by the formula in the statement of the theorem. Each factor, such as $A \vee B$, is positive (resp. negative) if and only if the lines A, B occur in the same (resp., opposite) order in the linear orders κ and λ, so χ_{ABC} is positive (resp. negative) if and only if the induced order of the lines A, B, C in κ and λ differ by an even (resp., odd) permutation. □

There is an intimate relation between stacked planar layouts and spindles. A stacked planar layout determines a vertical order of the lines over the plane of projection. If the lines can be transformed by an isotopy to a family of lines in distinct horizontal planes, without changing their stacking order, they can then be translated within those horizontal planes until their "crossing points" all coincide at a single point, a vertical line through which will then be the axis of a spindle. We use these ideas as the basis of a proof of the following theorem.

In a stacked planar layout, the lines appear to be "stacked" in an order "$<$", defined by $A < B$ if and only if the line A passes under the line B. The binary relation "$<$" is transitive, because, for any three lines A, B, C in a stacked planar layout, if $A < B$ and $B < C$, then $A < C$; otherwise the three lines would form a vortex. For any two lines A, B, either $A < B$ or $B < A$, so the relation "$<$" is that of a linear order.

Let $(A, B, \ldots, E)$ be the linear order of a set of lines, from "bottom" to "top" in a stacked planar layout over a plane which we shall assume, without loss of generality, to be the xy-plane. This means that at the "crossing point" of two lines L, M, the line L passes *under* the line M if and only if $L < M$ in the order "$<$". For any line L in the family, occurring in the i^{th} place in the order "$<$", select a *base point* $\beta(L)$ at the intersection of the line L with the horizontal plane Q_i, with equation $z = i$. Then assign to L a vector $\mu(L)$, its *velocity*, the vector obtained by projecting the oriented line segment $[L \wedge Q_i, L \wedge Q_{i+1}]$ into the xy-plane. Let L' be the projection of the line L into the xy-plane. For any point s on L', we have the corresponding parameter value $\alpha_L(s)$, defined by the equation

$$s = \beta'(L) + \alpha_L(s)\mu(L),$$

where $\beta'(L)$ is obtained by vertical projection of $\beta(L)$ onto the xy-plane. Over such a point $s \in L'$, the line has height $i + \alpha_L(s)$.

Theorem 57. *A configuration of n lines is isotopic to a stacked planar layout if and only if it is isotopic to a spindle.*

Proof: Let $\{A, \ldots, D\}$ be a family of n lines with a stacked planar layout, say, via vertical projection into the xy-plane. Assign to each line in the family its base point $\beta(L)$ and velocity $\mu(L)$, as above. Now, for every value of a real parameter k, $0 < k \leq 1$, and for every line L of the family, of rank i in the linear order "$<$" of the stacked planar layout, define a line L_k with base point $\beta(L_k) = \beta(L)$, velocity $\mu(L_k) = (1/k)\mu(L)$. In the limit as $k \to 0$, we have a horizontal line $L_0 = \beta(L) \vee \mu(L)$ in the plane Q_i. The lines $\{L_k; 0 \leq k \leq 1\}$ lie in the vertical plane through L, so for any two lines L, M in the family, the crossing point $s_k = L'_k \wedge M'_k$ does not change, and $s_k = s = L' \wedge M'$ for all k such that $0 \leq k \leq 1$. Assume $L < M$ in the stacking order "$<$". The heights of the lines L_k, M_k over their crossing point s in the layout are given by the formulas

$$i + \alpha_{L_k}(s) = i + k\alpha_L(s),$$

$$j + \alpha_{M_k}(s) = j + k\alpha_M(s).$$

Since L passes under M in the planar layout,

$$i + \alpha_L(s) < j + \alpha_M(s).$$

Since L occurs before M in the stacking order,

$$i < j.$$

These inequalities are preserved in convex linear combinations: for $0 \le k \le 1$,

$$k(i + \alpha_L(s)) + (1-k)i < k(j + \alpha_M(s)) + (1-k)j,$$

so

$$i + k\alpha_L(s) < j + k\alpha_M(s),$$

and the line L_k passes under the line M_k, for all parameter values $k, 0 \le k \le 1$. Using this isotopy, we know that any given family with stacked planar layout over the xy-plane is isotopic to a family of lines in distinct horizontal planes, in the same stacking order "$<$". Each line L is the join of its base point $\beta(L)$ with the direction vector $\mu(L)$, taken as a point at infinity in the xy-plane. Now translate all lines L within their planes Q_i, and parallel to themselves, until their base points $\beta(L)$ lie on the z-axis Z. This is an isotopy, because lines in parallel planes can meet each other only at infinity, that is, when they have the same direction. Finally, adjust the directions of the lines until they have slopes $i = 1, \ldots, n$ in the xy-plane, without changing the order of their slopes. The resulting family of lines is a spindle.

To prove the converse, we use the fact that the set of points in 3-space from which three given skew lines appear concurrent is a ruled surface of topological dimension 2; from most points in space the three lines will produce distinct "crossing points" for the three pairs of lines. Thus, viewed from almost any point off the z-axis Z, a spindle will yield a planar layout. Furthermore, from any such point above the plane $z = n$, the layout will be stacked, since the lines will appear to be stacked in the same order in which they are arranged along the vertical axis of the spindle. □

From the constructive isotopy in the proof of the previous theorem we learn how to obtain the (κ, λ)-code of an isotopic spindle immediately from the stacked planar layout. Perhaps the simplest technique is to use the increasing order of slopes $\{s_L;\ L \in D\}$, where $-\infty < s_L \le \infty$ as κ, and the (upwards) stacking order as λ.

Spindles which are distinct need not fall into distinct isotopy classes. In the following theorem, we give *three* transformations of spindles which preserve chiral signature, and we show that *two* of these can always be accomplished by isotopy.

By a *circular permutation* of a linearly ordered set we mean any iteration of the permutation which moves the last element to the first position in the order.[14] If a linear orders κ, κ' differ by a circular permutation, we write $\kappa \sim \kappa'$. By a *circular order* of a set we mean an equivalence class of linear orders obtained from one another by circular permutations. An (unordered) subset E is *consecutive* in a circular order (say, κ) of a set if and only if the elements in E occur consecutively in some linear order representing the circular order κ.

Local reversal of a subset E consecutive in a linear order κ is the permutation which rewrites the subset E in the opposite of the order induced by κ, within the interval it occupies in some appropriate linear order representing in κ. For instance, the linear order $BFCDAE$ is obtained from the order $BDCFAE$

[14] Not to be confused with *cyclic* permutations, which form a much larger class.

by local reversal of the subset $\{C, D, F\}$, while the circular order $ADCFBE$ is obtained by local reversal of the complementary set $\{A, B, E\}$ in the same order $BDCFAE$. This can perhaps be more easily seen in three steps:

$$BDCFAE \sim AEBDCF \rightarrow BEADCF \sim ADCFBE,$$

using circular permutations to make the set $\{A, B, E\}$ consecutive in the linear order.

Local exchange of a subset E consecutive in two circular orders κ, λ places the segment E in λ in its κ-order, the segment E in κ in its λ-order.

Theorem 58. *The following operations on spindles (κ, λ) preserve chiral signature:*

(1) circular permutations of κ and of λ,

(2) simultaneous local reversal of any subset that occurs consecutively in both circular orders (κ, λ),

(3) local exchange of any subset that occurs consecutively in both circular orders (κ, λ).

Proof: Two spindles $(\kappa, \lambda), (\kappa', \lambda')$ have the same chiral signature if and only if for every subset T consisting of three lines, the induced permutation $\kappa_T \rightarrow \kappa'_T$ has the same parity as the induced permutation $\lambda_T \rightarrow \lambda'_T$. Circular permutations perform *even* permutations of the induced order on any set of three elements, so they do not affect the chiral signature.

Let $E = \{A, B, C\}$ be a set of three lines occurring consecutively in both $\kappa_T \rightarrow \kappa'_T$. Local reversal of E performs an *odd* permutation on all triples having at least two elements in common with E, and this, in both κ and λ. Local exchange performs an *odd* permutation on *negative* chiral triples of (κ, λ) having at least two elements in common with E, and this, for both κ and λ. So both local operations preserve chiral signature. □

Conjecture 59. *Two spindles have the same chiral signature if and only if they are obtainable one from the other by a sequence of circular permutations, local reversals and local exchanges.*

In order to show that simultaneous local reversal of a subset occurring consecutively in the two orders of a spindle can be accomplished by an isotopy of a spindle, we need the following arithmetic and geometric lemmata.

Lemma 60. *If a, b, c, s, t are real numbers such that*

$$ab + st - c(at + bs) = 0, \text{ with } |a|, |b| < 1 < |s|, |t|,$$

then $1 < |c|$.

Proof: Let $f(a, b, c, s, t) = ab + st - c(at + bs)$. Since

$$f(-a, b, c, -s, t) = f(a, -b, c, s, -t) = -f(a, b, c, s, t),$$

we can, by simultaneously substituting (when necessary)

either $-a$ for a and $-s$ for s,

or $-b$ for b and $-t$ for t,

convert any instance where $f(a, b, c, s, t) = 0$ to another instance in which this is still true, but $a < s$ and $b < t$. In such a case, both $1 < s$ and $1 < t$, so $-1 < ab < 1 < st$ and $ab + st$ is positive. Since $a < s$ and $b < t$, $0 < (s - a)(t - b) = (ab + st) - (at + bs)$, and $at + bs < ab + st$. Likewise, since $0 < a + s$ and

$0 < b + t$, we have $0 < ab + at + bs + st$, and $-ab - st < at + bs$. Since $-(ab + st) < at + bs < (ab + st)$, we have

$$|at + bs| < |ab + st|.$$

But $f(a, b, c, s, t) = 0$, so $c = (ab + st)/(at + bs)$, and $|c| = |ab + st|/|at + bs| > 1$.[15] □

Lemma 61. *Let A be a line meeting the directrices H and K at points $(a, -1, 0, 0)$ and $(0, 0, b, 1)$, respectively, and such that $|a| < 1, |b| < 1$. Let B be a second line meeting the same directrices at points $(s, -1, 0, 0)$ and $(0, 0, t, 1)$, such that $1 < |s|, 1 < |t|$. Then the line A can be rotated freely about the y-axis, without meeting the line B.*

Proof: If the line A is rotated by an angle θ about the y-axis, the points $(a, -1, 0, 0)$ and $(0, 0, b, 1)$ move to points

$$(a\,cos\theta, -1, a\,sin\theta, 0),$$

$$(-b\,sin\theta, 0, b\,cos\theta, 1),$$

respectively. The rotated line A_θ has projective coordinates

$$\begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ -b\,sin\theta & ab & a\,cos\theta & -b\,cos\theta & -1 & a\,sin\theta \end{pmatrix}$$

while B has coordinates

$$\begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ 0 & st & s & -t & -1 & 0 \end{pmatrix}$$

So

$$A_\theta \vee B = ab - at\,cos\theta - bs\,cos\theta + st.$$

If the join $A_\theta \vee B$ were to be equal to zero for some value of θ, for $0 \le \theta \le 2\pi$, then by lemma 27, $|cos\theta| > 1$, a contradiction. So the lines A_θ and B never meet at a point. □

Theorem 62. *A spindle (κ, λ) is isotopic to any spindle (κ', λ') if the latter is obtained from the former by a sequence of any of the following operations:*

(1) independent circular permutations of κ and of λ;

(2) simultaneous local reversal of any subset which occurs consecutively in both linear orders κ, λ;

(3) global exchange of κ, λ.

Proof: To generate a circular permutation of κ, we simply perform a rotation of the spatial configuration around the z-axis. The corresponding isotopy for circular permutation of the order λ is a "rotation" about the line at infinity in the xy-plane: the top line of the stack moves vertically, passing through the plane at infinity, and becomes the bottom line of the stack, without changing the slope of its projection on the xy-plane.

Assume that a subset T of lines occurs concurrently in the linear orders κ, λ (but not necessarily in the same induced order). Move the points of contact along the directrices until the points of contact of lines in T lie in the intervals $(-1, +1)$ on both directrices, and all other points of contact lie outside those two intervals. This is possible without changing the order of the points of contact along the directrices, so the action performed is an isotopy. By Lemma 61, the lines in T can be simultaneously rotated by an angle π about the y-axis, without meeting any of the other lines. This rotation, which accomplishes the local inversion, is therefore an isotopy. □

[15] We wish to thank Carlos Klimann (INRIA) for his assistance with this proof.

While chiral signatures, orbits of diagrams under equivalence, Kauffman polynomials, etc., can only provide negative answers on the isotopy problem, the spindle moves of Theorem 62 enable us to construct an explicit isotopy between two line configurations (spindles).

16. Chiralities not realizable by spindles

In Figure 28 we show the graphs of all minimal non-spindle-realizable chiralities for up to 8 lines. For each minimal non-spindle-realizable chirality, we indicate all cousins, as isomorphism classes of unlabelled graphs.

The pentagon in the top row is the graph of a chiral signature on 6 lines. It is isomorphic to its own complement. As we know, this isomorphism of signatures does not carry over to an isotopy of lines. See configurations [6 – 10 – 10 – 4] and [6 – 10 – 10 – 5] in Appendix A (the catalogue). The complements of the two graphs on the left of the second row appear to the right in that same row. Corresponding graphs in the third and fourth rows are complementary. The same is the case for the fifth and sixth rows. The seventh row is arranged like the second row.

If any chiral signature on up to eight lines is not representable by a spindle, then it is *cousin to* a graph that contains an *induced subgraph* isomorphic to one of these 27 graphs.

The list of *all* minimal graphs whose chiralities are not spindle-representable is somewhat larger. For instance, the graph on the left of Figure 29 codes a signature which is not spindle representable, but not a minimal such signature. The given graph is cousin to those on the right, and they, in turn, contain the pentagon as induced subgraph.

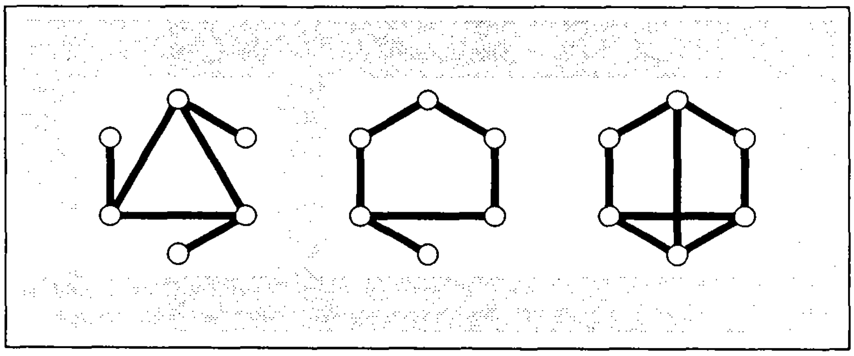


Figure 29. A graph cousin to graphs containing the pentagon as induced subgraph.

Theorem 63. *If the graph of a chiral signature contains an n-gon as induced subgraph, for $n \geq 5$, then it is not spindle representable.*

Proof: First we construct all possible spindle representations of the broken cycle consisting of $n-1$ vertices $2, 3, \ldots, n$ and edges $(2,3), (3,4), \ldots (n-1, n)$. Removing the symmetry due to circular permutations and global exchange, we assume without loss of generality that the permutations of the set $\{1,2,3\}$ induced by κ and λ are 123 and 132, respectively, thus satisfying the requirement that $(2,3)$ be an edge.

Since $(3,4)$ is an edge, but $(2,4)$ is not, 4 must be on the same side of 2 and on opposite sides of 3 in the linear orders $\begin{pmatrix} 123 \\ 132 \end{pmatrix}$. There are two possiblilites:

$$\begin{pmatrix} 1243 \\ 1324 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1423 \\ 1342 \end{pmatrix}$$

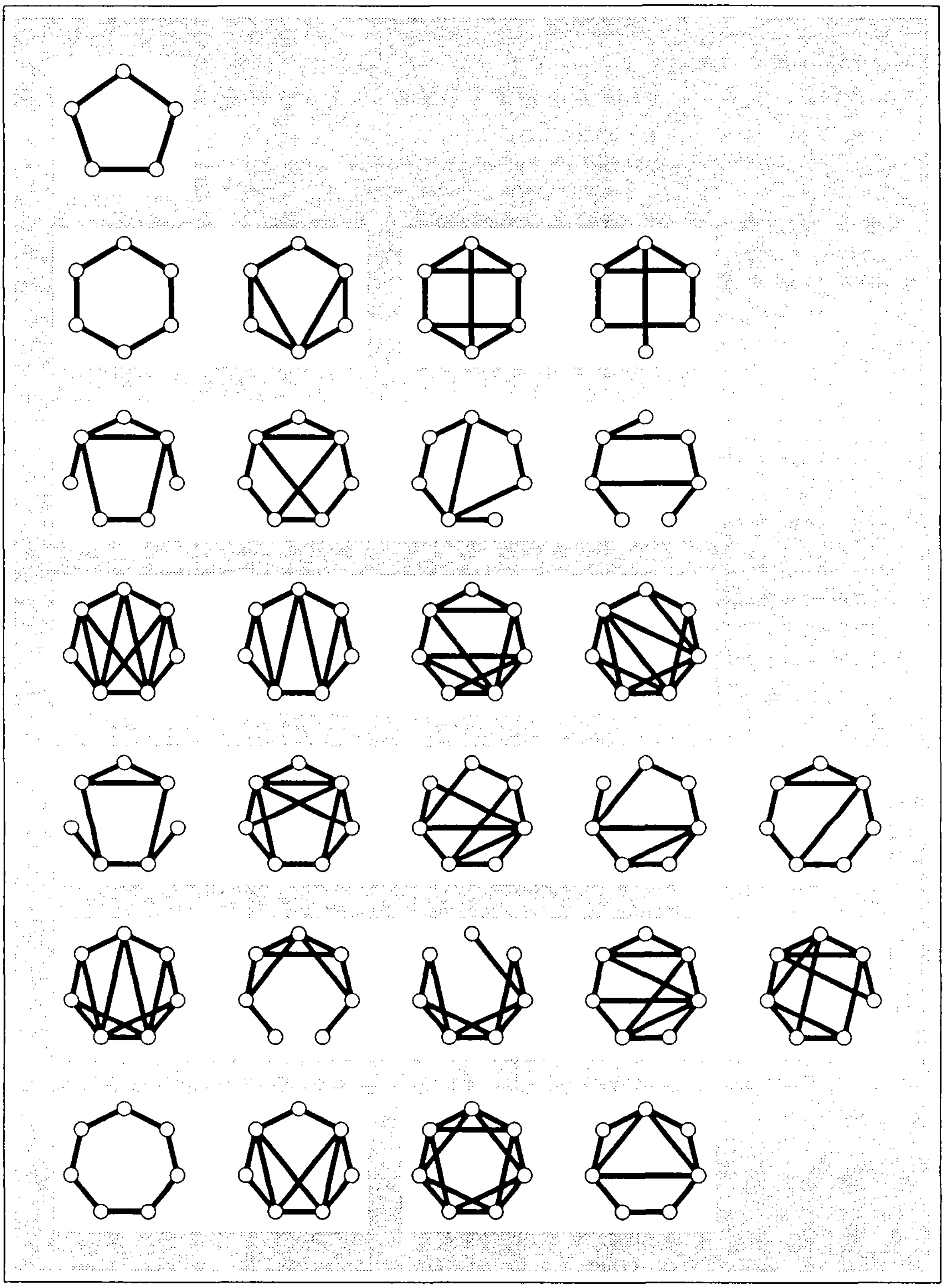


Figure 28. Isomorphism classes of non-spindle-representable chiralities.

These two solutions differ by global reversal followed by global exchange. So we need consider only the first case.

We prove by induction that the only representations of a broken n-cycle, for $n \geq 5$, are by the spindles

$$\begin{pmatrix} 124\ldots n-3, n, n-1 \\ 132547\ldots n-1, n-2, n \end{pmatrix}$$

for n even, $n \geq 6$, while

$$\begin{pmatrix} 124\ldots n-1, n-2, n \\ 132\ldots n-3, n, n-1 \end{pmatrix}$$

for n odd, $n \geq 5$, the uniqueness being up to global reversal, global exchange, and circular permutations.

The arguments in the three cases $4 \to 5$ and $n \to n+1$, for n even or odd, are essentially the same. We give the proof for the case n even. We are looking for a representation of the broken $n+1$-cycle which extends the representation

$$(\kappa, \lambda) = \begin{pmatrix} 124\ldots n-3, n, n-1 \\ 132\ldots n-1, n-2, n \end{pmatrix}$$

of the broken n-cycle. In such an extension, if $n+1$ were to be placed before $n-1$ in κ, it would have to be placed before $n-1$ in λ, because $(n-1, n+1)$ is not an edge. It would then be before $n-2$ in λ, so would have to be before $n-2$ in κ, $(n-2, n+1)$ not being an edge. But then $n+1$ is before n in both permutations, contradicting the fact that $(n, n+1)$ is an edge. So $n+1$ must come at the end of the permutation κ, and just before the n in λ.

We now show that none of these spindles is extendable to a representation of the n-cycle, $n \geq 5$ by addition of a line $n+1$. Again, the arguments needed for the cases n even and odd are essentially the same. We give the argument for n even, assuming that

$$(\kappa, \lambda) = \begin{pmatrix} 124\ldots n-3, n, n-1 \\ 132\ldots n-1, n-2, n \end{pmatrix}$$

is the spindle which results by dropping the line $n+1$. The edges to be added are $(2, n+1)$ and $(n, n+1)$.

If $n+1$ is placed before 2 in κ, it must be placed after 2 in λ. Since it is then after 3 in λ, it must be placed after 3 in κ, in contradiction to our assumption that $n+1$ is before 2 in that permutation.

If $n+1$ is placed after 2 in κ, it must be placed before 2 in λ. Since it is then before 4 in λ, it must be placed before 4 in κ. But then $n+1$ is before n in both κ and λ, in contradiction to our assumption that $n, n+1$ is an edge. □

Theorem 64. *There are at least four infinite classes of graphs of minimal non-spindle-representable chiralities: the n-cycles for $n \geq 5$, their complements, their (unique) cousins, and the complements of those cousins. These four classes have a single graph in common: the pentagon.*

Proof: If a graph is representable by a spindle (κ, λ), its complement is representable by (κ, λ^{opp}), and their cousins are representable by those same spindles. Thus if a graph is that of a minimal non-spindle-representable chirality, the same holds for its complement, and for the cousins of those two graphs. By the symmetry of the n-cycle, it has only one cousin, up to unlabelled graph isomorphism. And the same holds for its complement. In the proof of Theorem 63, we showed not only that the n-cycle is not spindle representable, for $n \geq 5$, but also that all proper subgraphs (the broken n-cycles) *are* so representable. □

The third and fourth rows of Figure 28 show the cousins of the *turtle*, the first example in row 3, and their respective complements. The fifth and sixth rows show the cousins and their complements for the *frog*, the first example in row 5. The four chiral signatures represented by the turtle, the frog, and their complements, are the *only* examples (up to eight lines) of minimal non-spindle-representable chiralities falling outside the classes listed in Theorem 64.

Let T be the class of graphs which are comparability graphs [Ga] of partially-ordered sets of Dushnik-Miller dimension ≤ 2. A graph $G = (V, E)$ is in the class T if and only if there exist two linear orders α and β on the vertex set V of G such that

$$(a, b) \in E \iff (a <_\kappa b \Leftrightarrow a <_\lambda b)$$

that is, that edge is in the comparability graph if and only if the pair is ordered in the partial order $\alpha \cap \beta$, if and only if the orders α and β agree on the pair a, b.

Proposition 65. *A graph is spindle-representable if and only if it is in the class T.*

Proof: A graph G on $n-1$ vertices $v_2, \ldots v_n$ is spindle-representable if and only if there are two linear orders κ and λ on the set $\{2, \ldots, n\}$ such that v_i, v_j is an edge of G if and only if the linear orders κ, λ *disagree* on the pair $\{i, j\}$. In this event, the intersection of the linear orders κ, λ^{opp} provide a partial order of Dushnik-Miller dimension ≤ 2, for which G is the comparability graph, and conversely. □

As Kelly and Trotter [Ke_2] have observed, a graph G is in the class T if and only if both G and its complement are comparability graphs of partially-ordered sets. The graphs not representable as comparability graphs, and minimal in the order: $G \leq H$ if and only if G is an induced subgraph of H, have been listed by Gallai [Ga],[Tr_1]. Except for ten exceptional cases, they fall into eight infinite families.

Proposition 66. *A graph G is the graph of an abstract chirality minimally non-spindle-representable if and only it or its complement occurs in Gallai's list, but neither G, its complement, nor their cousins have a proper induced subgraph in that list.*

Proof: If a graph G is minimally not-spindle-representable, then it is not the case that both G and its complement are comparability graphs. One of those graphs will be a minimal graph not representable as a comparability graph, and will be in Gallai's list. If the corresponding abstract chiral signature is minimally non-spindle-representable, then no cousin of the graph G can have a proper induced subgraph in Gallai's list. □

17. Open problems and related research

The main as-yet-unsolved problems in the area include

(1) To find a pair of unlabelled configurations of projective lines which have the same Kauffman polynomial, but which are not isotopic.

(2) To find a pair of configurations of projective lines which are isotopic as pseudoline configurations, but are not isotopic.

(3) To find a geometric invariant (bracket polynomial) whose sign distinguishes between the three pairs of configurations shown in Figure 26.

(4) To find the smallest example of an abstract chiral signature (satisfying the "rule of four"), which is not representable as the signature of a line configuration. (The smallest known example has 71 lines (P. Shor, unpublished).

(5) To find any additional general relations holding among the signs of all chiral signatures, other than the "rule of four".

(6) The use of projective braids (the single braid α, without its opposite $\hat{\alpha}$, in Section 12) seems to open promising avenues for further research on pseudoline diagrams. Find a canonical form and algorithm for reduction to canonical form for projective braid representations of line configurations (cf. [Mo_2]).

(7) Prove or disprove Conjecture 61, that two spindles have the same chiral signature if and only if they are obtainable one from the other by a sequence of circular permutations, local reversals and local exchanges.

(8) Find a natural isotopy construction for local exchange. It is known [Pe_{4-6}] that this can be accomplished by diagram moves. At first sight, an isotopy seems impossible: the required rotation causes most pairs of lines to cross. But recent claims (two paragraphs, below) that spindles with the same chirality are isotopic would indicate that the required isotopy exists. To test these ideas, consider two spindles on eight lines

$$\begin{pmatrix} 12345678 \\ 31427586 \end{pmatrix} \rightarrow \text{(local exchange)} \rightarrow \begin{pmatrix} 12347586 \\ 31425678 \end{pmatrix}$$

which differ by a local exchange of 4 elements, and where the exchange cannot be accomplished by a sequence of moves listed in Theorem 64.

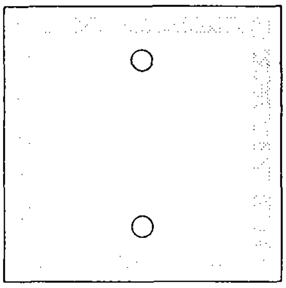
Some further work has already appeared in pre-publication format, by one of the present authors, [Pe_4]. The diagram moves of Section 9 can be used to define an equivalence relation on pseudoline diagrams (pseudoline arrangements with a crossing relation under). This relaxation of isotopy of line configurations provides a useful framework in which to investigate the *combinatorics* of line configurations, setting aside the many difficulties inherent in their *geometry*. In Section 14 we showed how the Kauffman polynomial appears in a natural way as an invariant under diagram moves. The notion of chirality can also be defined combinatorially for pseudoline diagrams, where it also turns out to be an invariant of diagram moves.

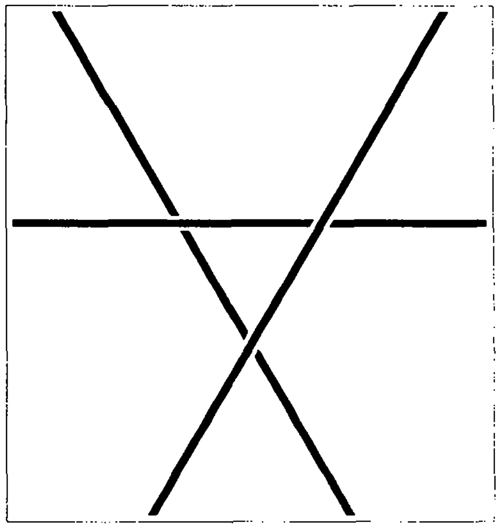
Likewise proven in [Pe_4] is that spindles with the same chiral signature are always isotopic as pseudoline diagrams, and are hence flexibly isotopic, by Theorem 50. In a private communication, S. Hashin and V. Mazurovskii claim to have sharpened this theorem to obtain an isotopy of line configurations linking such spindles.

In contrast to the general knot isotopy problem, the equivalence of diagrams under moves is decidable. This is proven in [Pe_4] by passing to an equivalent word problem with four types of moves which do not change the word length. It is still an open problem, however, to decide the equivalence of diagrams in an efficient way.

Appendix A

18. Table of Kauffman polynomials for up to 6 lines

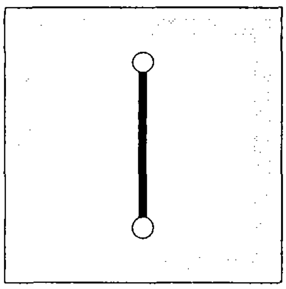


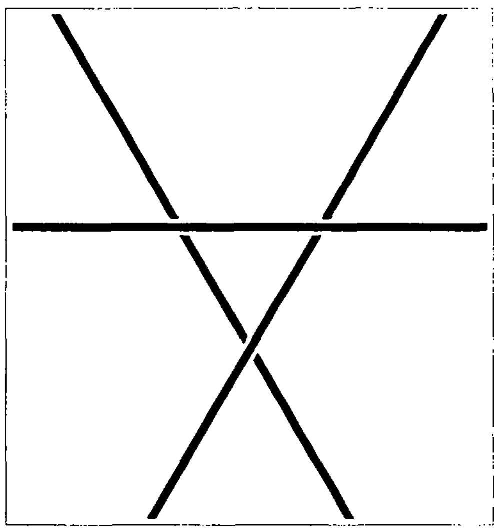


[3-1-0]

Three lines with positive chirality, spindle ((123)(123))

$$-2A^3 - 1A^1B^2 - 1A^{-1}B^4$$

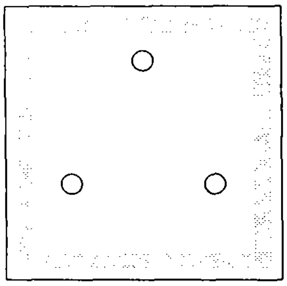


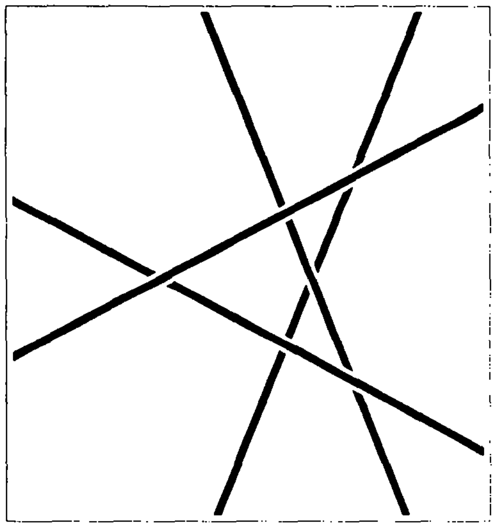


[3-0-1]

Three lines with negative chirality, spindle ((123)(132))

$$-1A^4B^{-1} - 1A^2B^1 - 2B^3$$

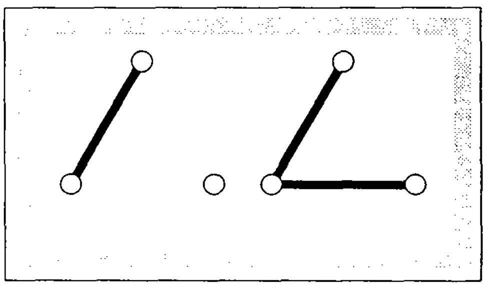


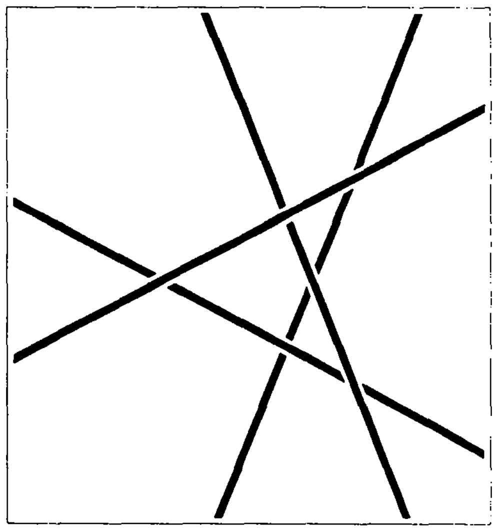


[4-4-0]

Four lines with positive chiral signature, spindle ((1234)(1234))

$$-2A^5B^1 - 3A^4B^2 - 1A^3B^3 - 1B^6 - 1A^{-1}B^7$$

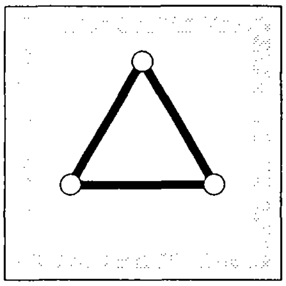


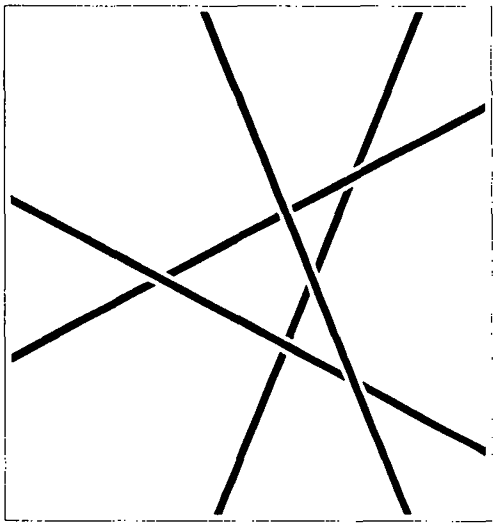


[4-2-2]

Four lines with mixed chiral signature, spindle ((1234)(1324))

$$-1A^6 - 2A^5B^1 - 1A^4B^2 - 1A^2B^4 - 2A^1B^5 - 1B^6$$

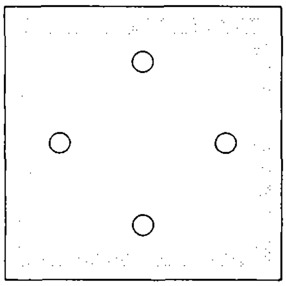


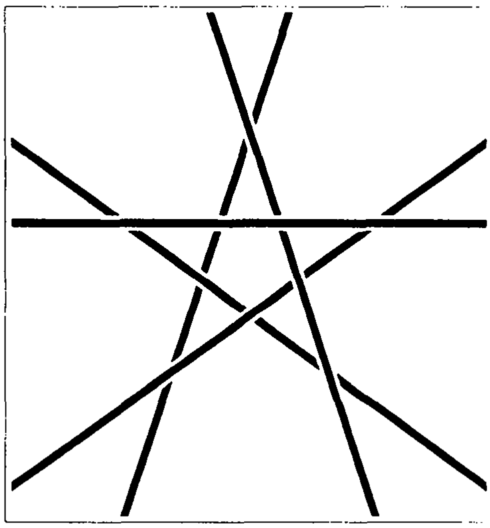


[4-0-4]

Four lines with negative chiral signature, spindle ((1234)(1432))

$-1A^7B^{-1} - 1A^6 - 1A^3B^3 - 3A^2B^4 - 2A^1B^5$

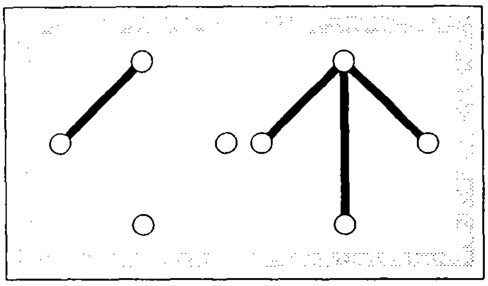


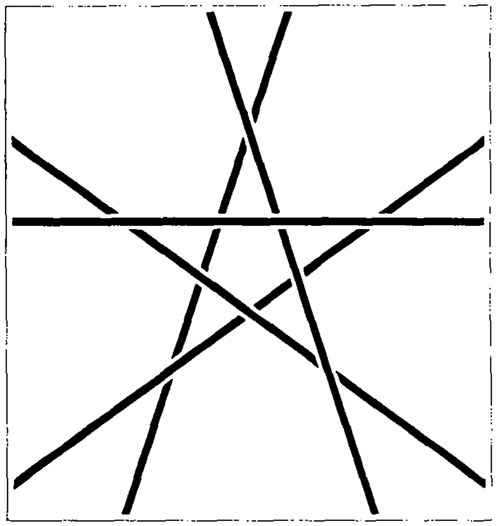


[5-10-0]

Five lines with positive chiral signature, spindle ((12345)(12345))

$+5A^8B^2 + 4A^6B^4 + 4A^4B^6 + 1A^2B^8 + 1B^{10} + 1A^{-2}B^{12}$

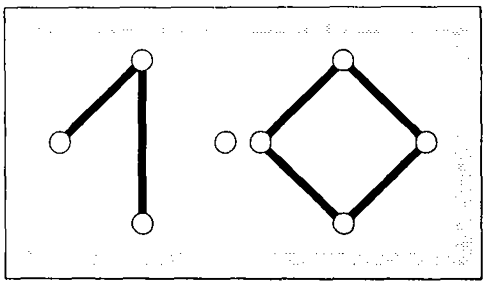


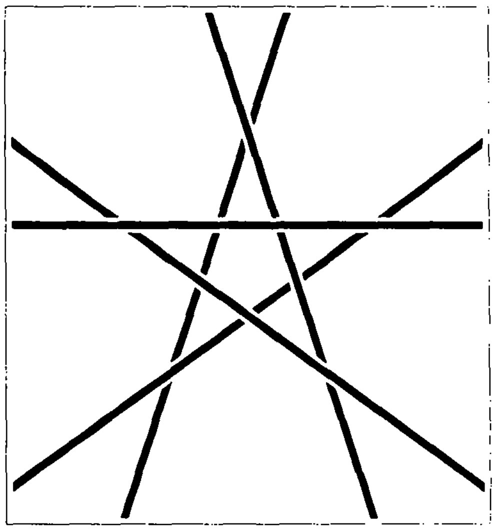


[5-7-3]

Five lines with chiral signature (7+, 3−), spindle ((12345)(13245))

$$+3A^9B^1 + 3A^7B^3 + 5A^5B^5 + 2A^3B^7 + 2A^1B^9 + 1A^{-1}B^{11}$$

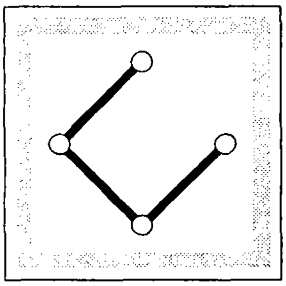


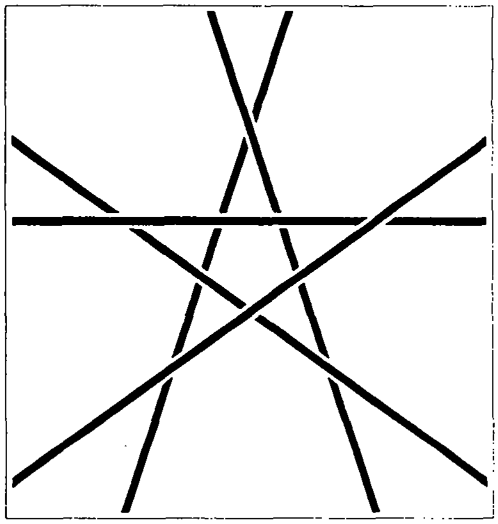


[5-6-4]

Five lines with chiral signature (6+, 4−), spindle ((12345)(14523))

$$+1A^{10} + 4A^8B^2 + 3A^6B^4 + 4A^4B^6 + 2A^2B^8 + 2B^{10}$$

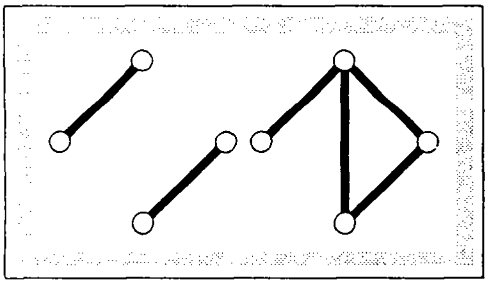


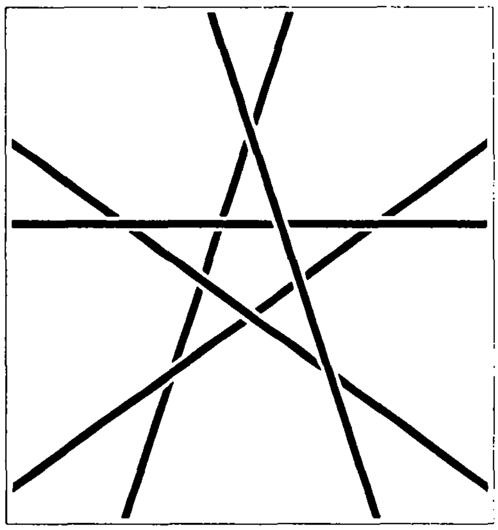


[5-5-5]

Five lines with chiral signature (5+, 5−), spindle ((12345)(14253))

$$+4A^9B^1 + 1A^7B^3 + 6A^5B^5 + 1A^3B^7 + 4A^1B^9$$

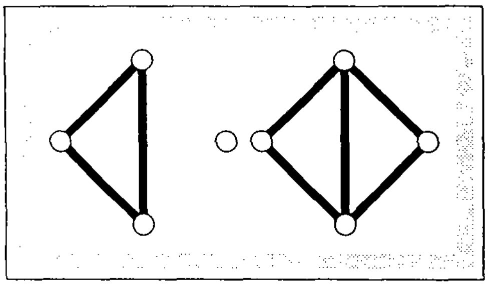


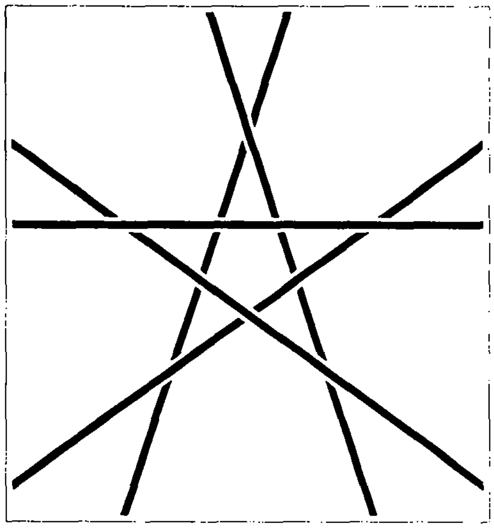


[5-4-6]

Five lines with chiral signature (4+, 6−), spindle ((12345)(13254))

$$+2A^{10} + 2A^8B^2 + 4A^6B^4 + 3A^4B^6 + 4A^2B^8 + 1B^{10}$$

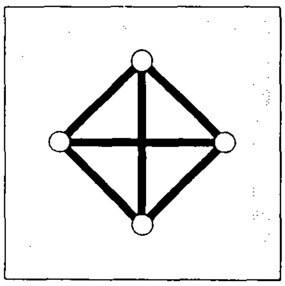


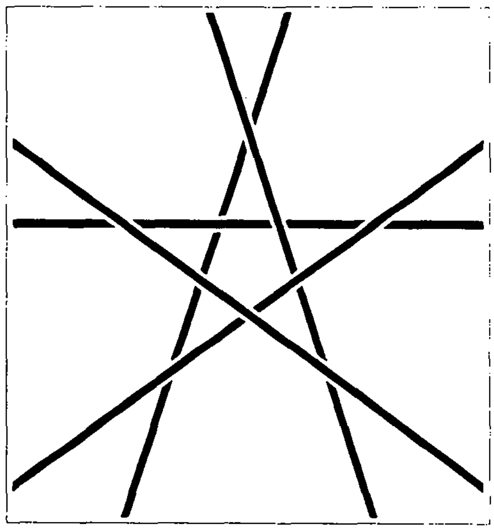


[5-3-7]

Five lines with chiral signature (3+, 7−), spindle ((12345)(14325))

$$+1A^{11}B^{-1} + 2A^{9}B^{1} + 2A^{7}B^{3} + 5A^{5}B^{5} + 3A^{3}B^{7} + 3A^{1}B^{9}$$

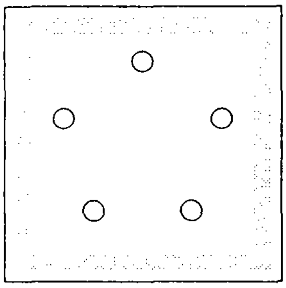


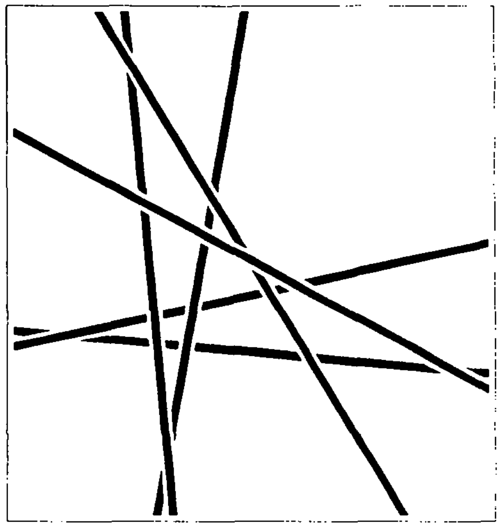


[5-0-10]

Five lines with negative chiral signature, spindle ((12345)(15432))

$$+1A^{12}B^{-2} + 1A^{10} + 1A^{8}B^{2} + 4A^{6}B^{4} + 4A^{4}B^{6} + 5A^{2}B^{8}$$

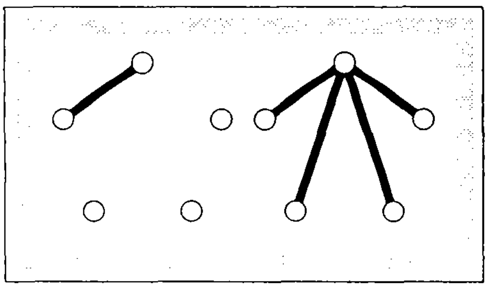


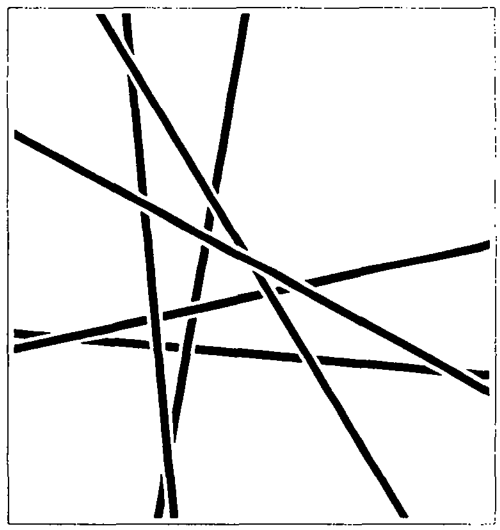


[6-20-0]

Six lines, chirality all positive, spindle (123456)(123456)

$$+5A^{11}B^{4}+9A^{10}B^{5}+4A^{9}B^{6}+1A^{7}B^{8}+5A^{6}B^{9}+4A^{5}B^{10}$$
$$+1A^{3}B^{12}+1A^{2}B^{13}+1A^{-1}B^{16}+1A^{-2}B^{17}$$

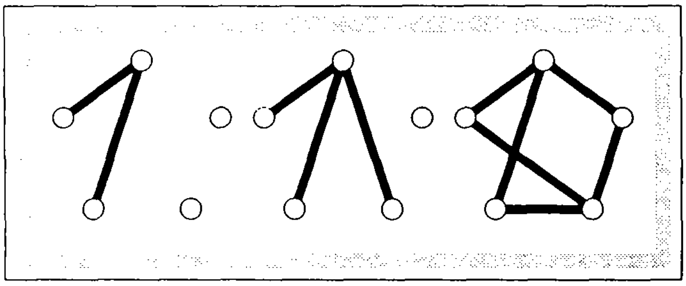




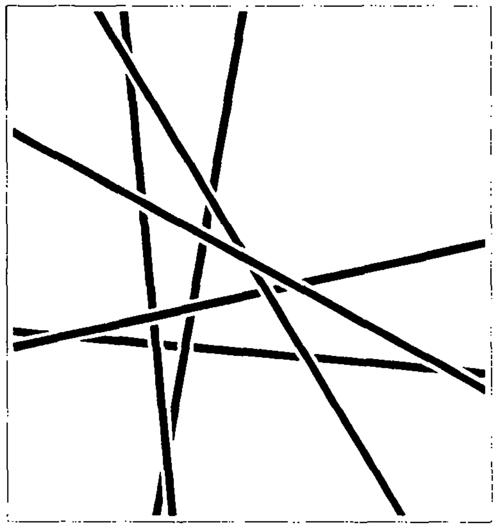
[6-16-4]

Six lines, chirality (16+, 4−), spindle (123456)(132456)

$$+3A^{12}B^{3}+6A^{11}B^{4}+3A^{10}B^{5}+3A^{8}B^{7}+7A^{7}B^{8}+4A^{6}B^{9}$$
$$+1A^{4}B^{11}+2A^{3}B^{12}+1A^{2}B^{13}+1B^{15}+1A^{-1}B^{16}$$

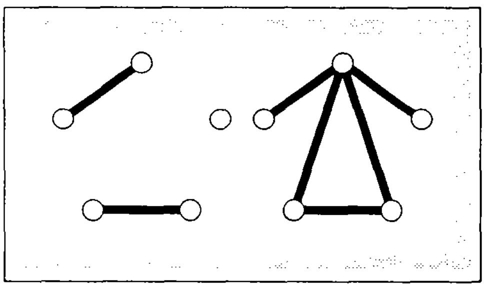


[6-14-6]

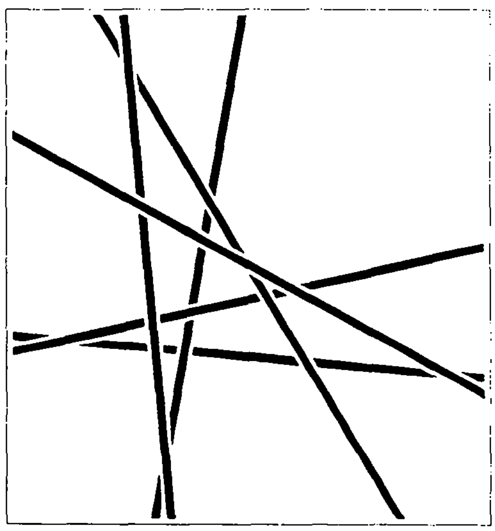


Six lines, chirality (14+, 6−), spindle (123456)(134256)

$$+1A^{13}B^2 + 3A^{12}B^3 + 4A^{11}B^4 + 3A^{10}B^5 + 2A^9B^6 + 3A^8B^7$$
$$+4A^7B^8 + 4A^6B^9 + 3A^5B^{10} + 1A^4B^{11} + 1A^2B^{13} + 2A^1B^{14}$$
$$+1B^{15}$$

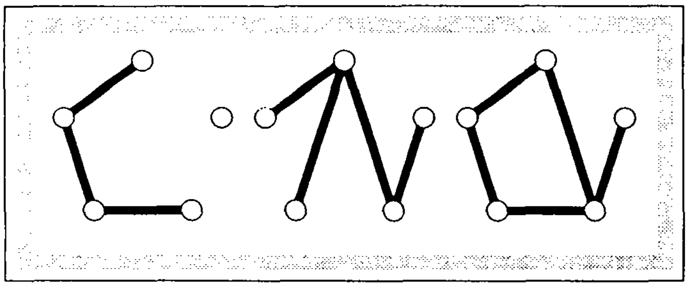


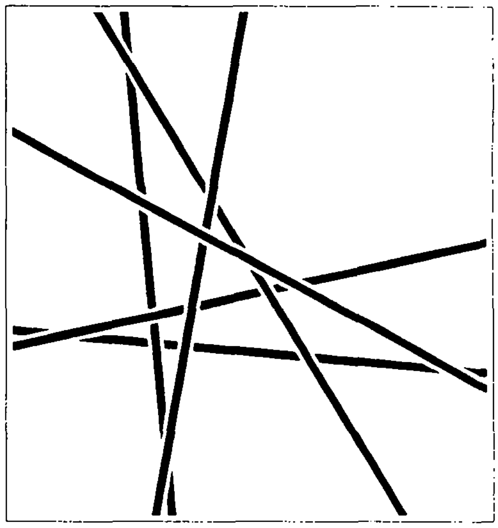
[6-12-8-1]



Six lines, chirality (12+, 8−), spindle (123456)(132546)

$$+2A^{13}B^2 + 4A^{12}B^3 + 2A^{11}B^4 + 3A^9B^6 + 7A^8B^7 + 4A^7B^8$$
$$+2A^5B^{10} + 4A^4B^{11} + 2A^3B^{12} + 1A^1B^{14} + 1B^{15}$$

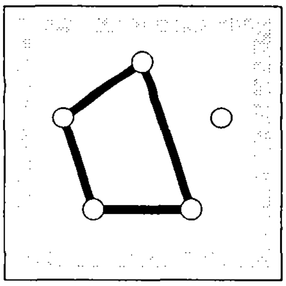


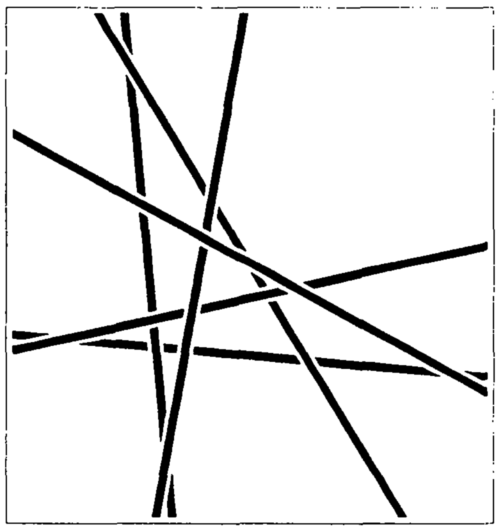


[6-12-8-2]

Six lines, chirality (12+, 8−), spindle (123456)(142536)

$$+1A^{13}B^{2}+4A^{12}B^{3}+5A^{11}B^{4}+1A^{10}B^{5}-1A^{9}B^{6}+4A^{8}B^{7}$$
$$+8A^{7}B^{8}+4A^{6}B^{9}-1A^{5}B^{10}+3A^{3}B^{12}+3A^{2}B^{13}+1A^{1}B^{14}$$

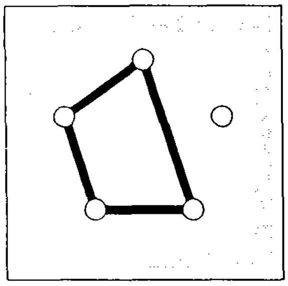


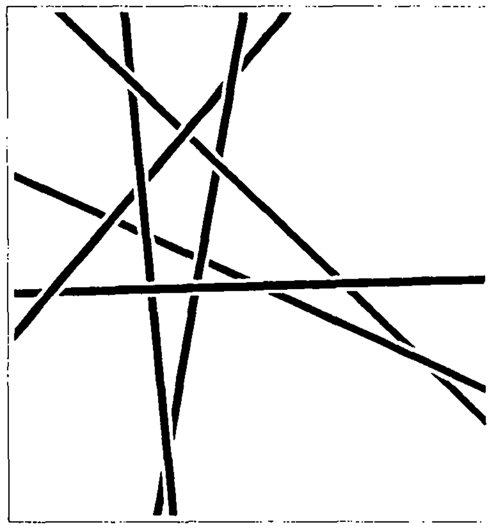


[6-12-8-3]

Six lines, chirality (12+, 8−), spindle (123456)(145236)

$$+1A^{14}B^{1}+1A^{13}B^{2}+4A^{11}B^{4}+7A^{10}B^{5}+3A^{9}B^{6}+2A^{7}B^{8}$$
$$+5A^{6}B^{9}+3A^{5}B^{10}+2A^{3}B^{12}+3A^{2}B^{13}+1A^{1}B^{14}$$

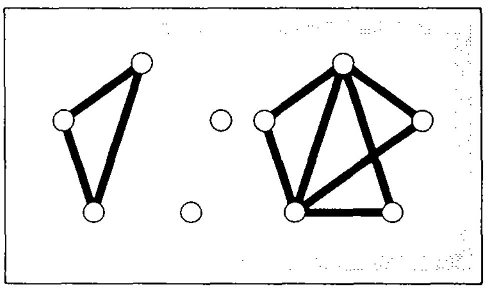


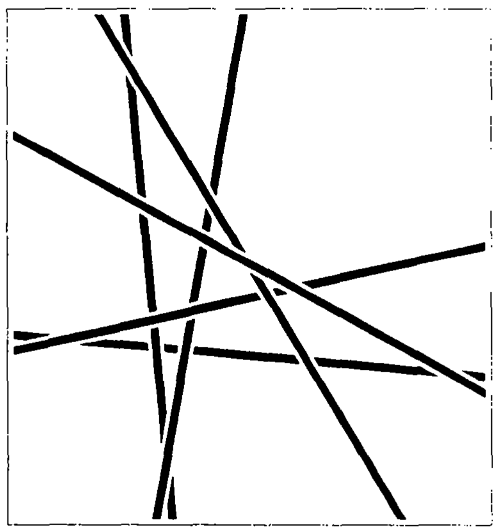


[6-12-8-4]

Six lines (Mazurovskii's L), no spindle representation

$$+1A^{16}B^{-1} - 5A^{14}B^{1} + 15A^{12}B^{3} + 10A^{11}B^{4} - 13A^{10}B^{5} \\ -12A^{9}B^{6} + 15A^{8}B^{7} + 22A^{7}B^{8} - 1A^{6}B^{9} - 12A^{5}B^{10} \\ +1A^{4}B^{11} + 8A^{3}B^{12} + 3A^{2}B^{13}$$

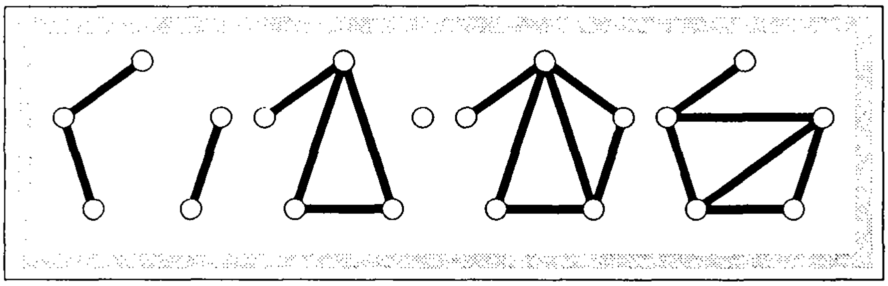


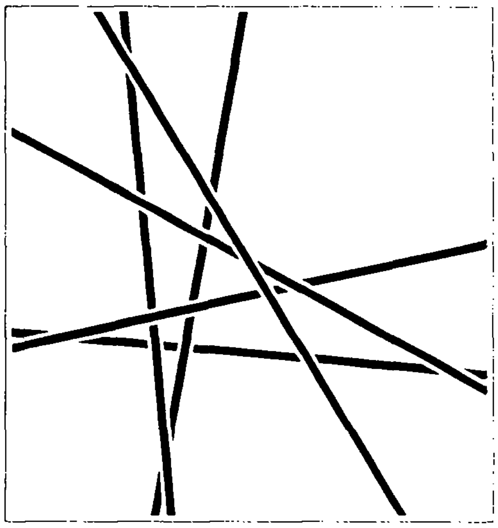


[6-10-10-1]

Six lines, chirality (10+, 10−), spindle (123456)(143256)

$$+1A^{14}B^{1} + 3A^{13}B^{2} + 2A^{12}B^{3} + 1A^{10}B^{5} + 3A^{9}B^{6} + 6A^{8}B^{7} \\ +6A^{7}B^{8} + 3A^{6}B^{9} + 1A^{5}B^{10} + 2A^{3}B^{12} + 3A^{2}B^{13} + 1A^{1}B^{14}$$

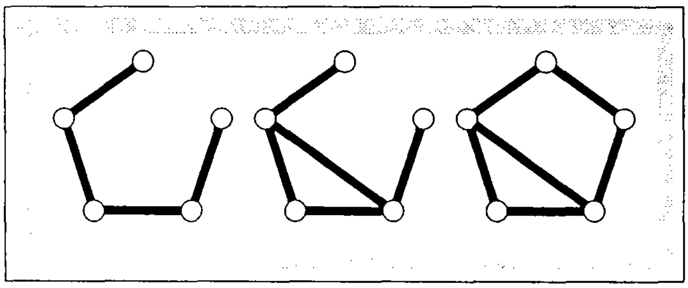


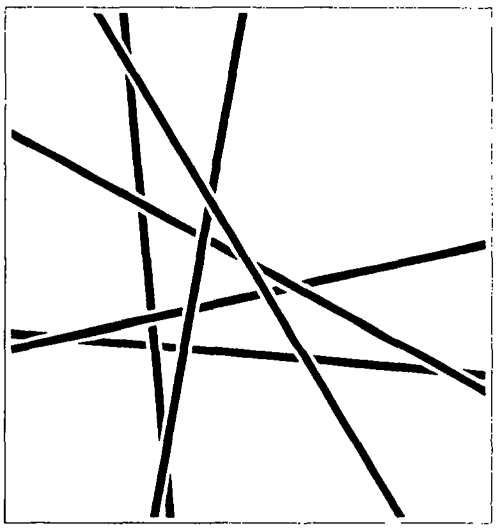


[6-10-10-2]

Six lines, chirality (10+, 10−), spindle (123456)(134265)

$$+1A^{14}B^{1}+2A^{13}B^{2}+2A^{12}B^{3}+2A^{11}B^{4}+2A^{10}B^{5}$$
$$+3A^{9}B^{6}+4A^{8}B^{7}+4A^{7}B^{8}+3A^{6}B^{9}+2A^{5}B^{10}$$
$$+2A^{4}B^{11}+2A^{3}B^{12}+2A^{2}B^{13}+1A^{1}B^{14}$$

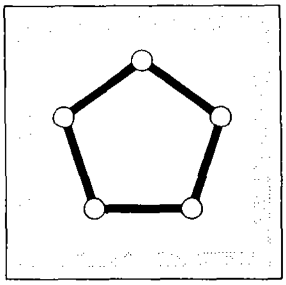


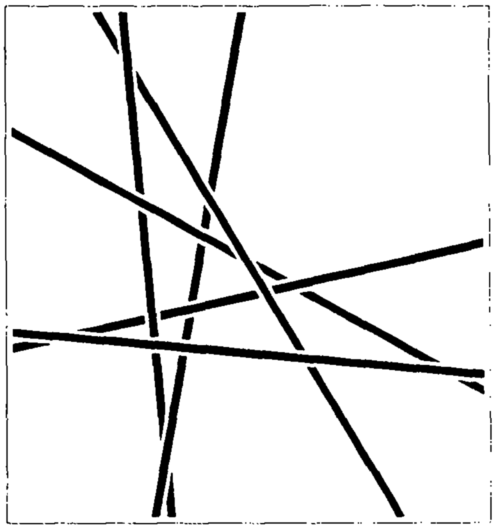


[6-10-10-3]

Six lines, chirality (10+, 10−), spindle (123456)(142635)

$$+2A^{13}B^{2}+5A^{12}B^{3}+3A^{11}B^{4}-2A^{10}B^{5}+8A^{8}B^{7}+8A^{7}B^{8}$$
$$-2A^{5}B^{10}+3A^{4}B^{11}+5A^{3}B^{12}+2A^{2}B^{13}$$

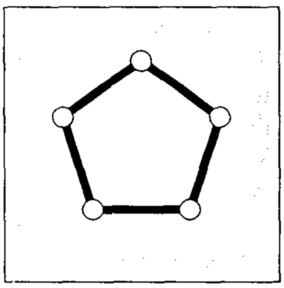


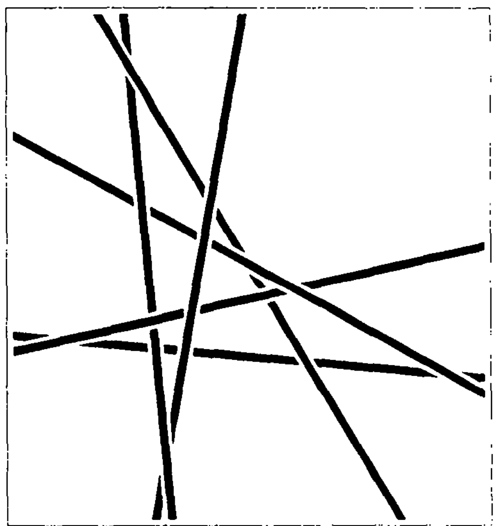


[6-10-10-4]

Six lines, chirality $(10+, 10-)$, no spindle representation

$$+5A^{12}B^{3} + 10A^{11}B^{4} - 10A^{9}B^{6} + 1A^{8}B^{7} + 16A^{7}B^{8} \\ +10A^{6}B^{9} - 6A^{5}B^{10} - 5A^{4}B^{11} + 6A^{3}B^{12} \\ +6A^{2}B^{13} - 1B^{15}$$

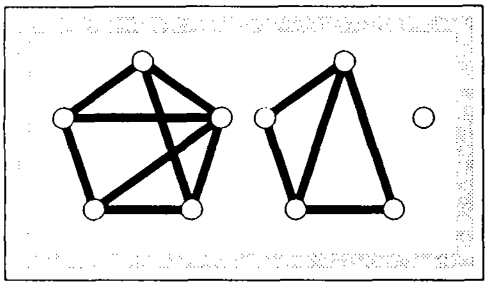


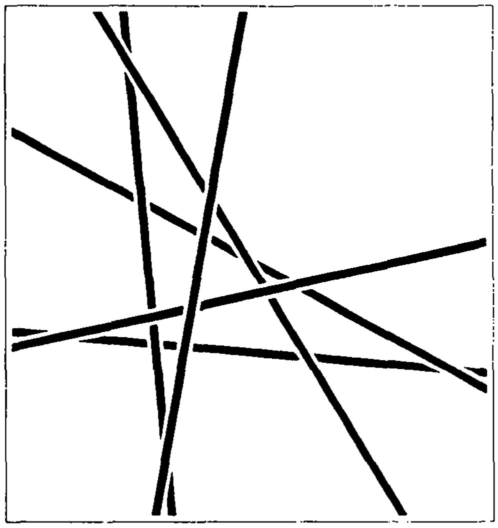


[6-10-10-5]

Six lines, chirality $(10+, 10-)$, no spindle representation

$$-1A^{15} + 6A^{13}B^{2} + 6A^{12}B^{3} - 5A^{11}B^{4} - 6A^{10}B^{5} + 10A^{9}B^{6} \\ +16A^{8}B^{7} + 1A^{7}B^{8} - 10A^{6}B^{9} + 10A^{4}B^{11} + 5A^{3}B^{12}$$

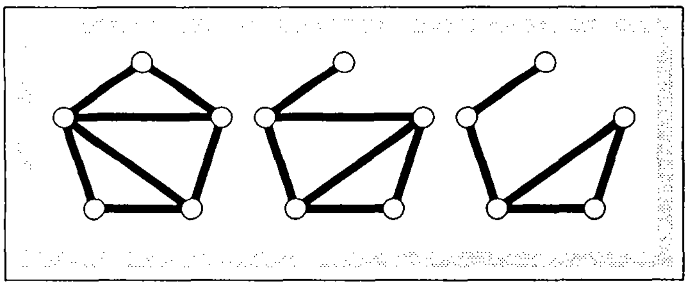


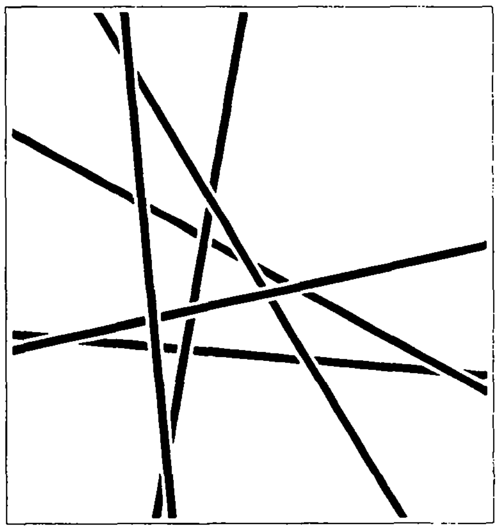


[6-8-12-1]

Six lines, chirality (8+, 12−), spindle (123456)(164523)

$$+1A^{15}+1A^{14}B^{1}+2A^{12}B^{3}+4A^{11}B^{4}+2A^{10}B^{5}+4A^{8}B^{7}$$
$$+7A^{7}B^{8}+3A^{6}B^{9}+2A^{4}B^{11}+4A^{3}B^{12}+2A^{2}B^{13}$$

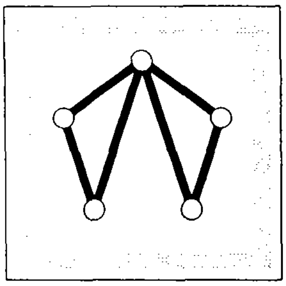


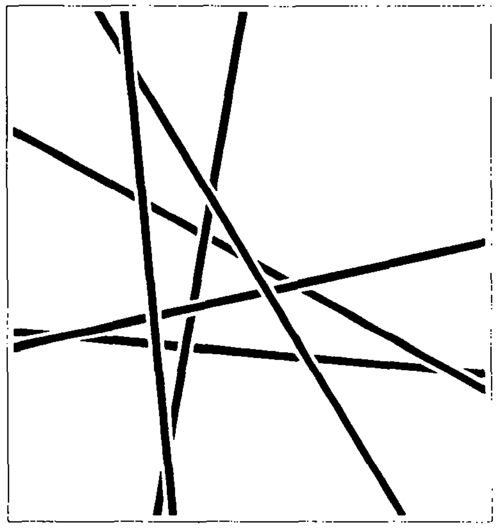


[6-8-12-2]

Six lines, chirality (8+, 12−), spindle (123456)(163524)

$$+1A^{14}B^{1}+3A^{13}B^{2}+3A^{12}B^{3}-1A^{10}B^{5}+4A^{9}B^{6}+8A^{8}B^{7}$$
$$+4A^{7}B^{8}-1A^{6}B^{9}+1A^{5}B^{10}+5A^{4}B^{11}+4A^{3}B^{12}+1A^{2}B^{13}$$

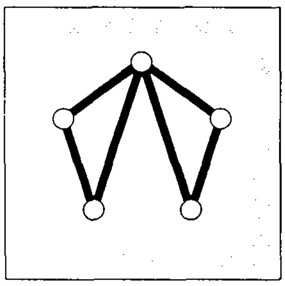


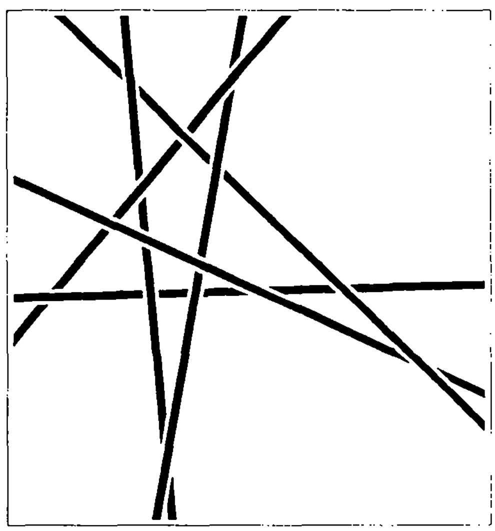


[6-8-12-3]

Six lines, chirality (8+, 12−), spindle (123456)(163254)

$$+1A^{14}B^{1}+3A^{13}B^{2}+2A^{12}B^{3}+3A^{10}B^{5}+5A^{9}B^{6}+2A^{8}B^{7}$$
$$+3A^{6}B^{9}+7A^{5}B^{10}+4A^{4}B^{11}+1A^{2}B^{13}+1A^{1}B^{14}$$

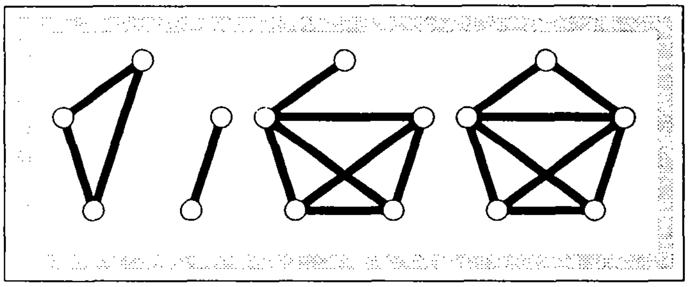


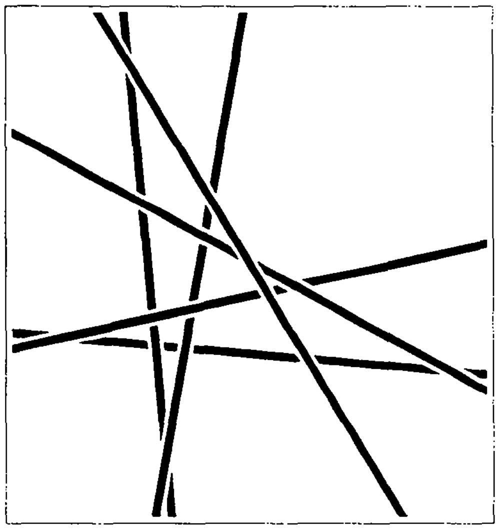


[6-8-12-4]

Six lines (Mazurovskii's L'), no spindle representation

$$+3A^{13}B^{2}+8A^{12}B^{3}+1A^{11}B^{4}-12A^{10}B^{5}-1A^{9}B^{6}$$
$$+22A^{8}B^{7}+15A^{7}B^{8}-12A^{6}B^{9}-13A^{5}B^{10}+10A^{4}B^{11}$$
$$+15A^{3}B^{12}-5A^{1}B^{14}+1A^{-1}B^{16}$$

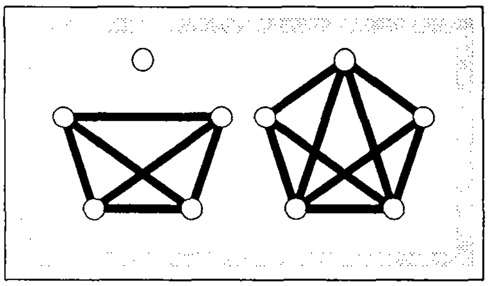


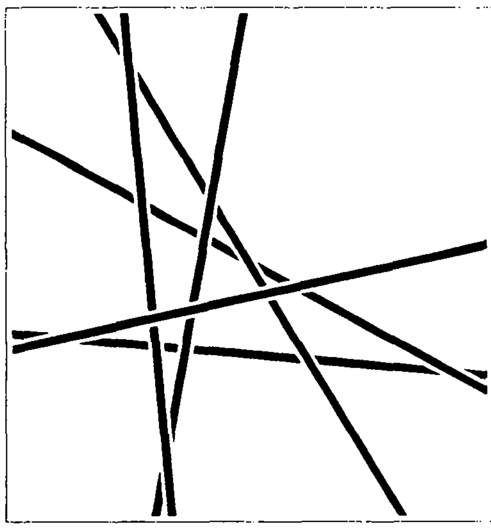


[6-6-14]

Six lines, chirality (6+, 14−), spindle (123456)(143256)

$$+1A^{15}+2A^{14}B^{1}+1A^{13}B^{2}+1A^{11}B^{4}+3A^{10}B^{5}+4A^{9}B^{6}$$
$$+4A^{8}B^{7}+3A^{7}B^{8}+2A^{6}B^{9}+3A^{5}B^{10}+4A^{4}B^{11}+3A^{3}B^{12}$$
$$+1A^{2}B^{13}$$

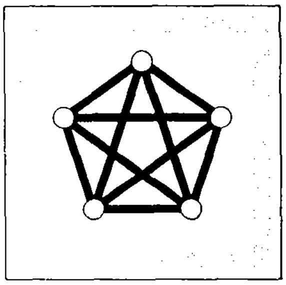


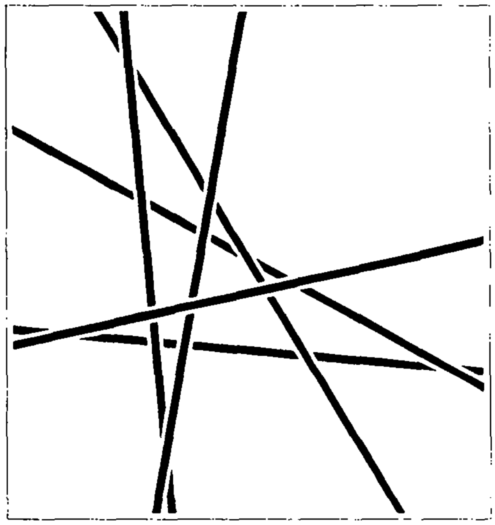


[6-4-16]

Six lines, chirality (4+, 16−), spindle (123456)(165342)

$$+1A^{16}B^{-1}+1A^{15}+1A^{13}B^{2}+2A^{12}B^{3}+1A^{11}B^{4}+4A^{9}B^{6}$$
$$+7A^{8}B^{7}+3A^{7}B^{8}+3A^{5}B^{10}+6A^{4}B^{11}+3A^{3}B^{12}$$





[6-0-20]

Six lines, all negative, spindle (123456)(165432)

$$+1A^{17}B^{-2} + 1A^{16}B^{-1} + 1A^{13}B^{2} + 1A^{12}B^{3} + 4A^{10}B^{5}$$
$$+5A^{9}B^{6} + 1A^{8}B^{7} + 4A^{6}B^{9} + 9A^{5}B^{10} + 5A^{4}B^{11}$$

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